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THE COMPRESSIBLE LAMINAR BOUNDARY LAYER WITH HEAT TRANSFER AND SMALL PRESSURE GRADIENT

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SUMMARY

A perturbation method for the calculation of velocity and temperature profiles and skin-friction and heat-transfer characteristics for two-dimensional, compressible laminar boundary layers with heat transfer and a small arbitrary pressure gradient is presented. The permissible pressure gradients include those of a form and magnitude usually encountered over slender aerodynamic shapes in supersonic flight. The method applies for any constant Prandtl number, but results, aside from special examples, are presented for a Prandtl number of 0.72. For the case of heat transfer, the wall temperature is assumed constant.

A large number of universal functions are given in tabular form, so that the amount of effort required in a practical application is reduced to the arithmetic combination of several tabulated values. The computation procedure is summarized in a section entitled "APPLICATION OF ANALYSIS."

The combined effects of heat transfer and pressure gradient on boundary-layer characteristics are demonstrated by applying the results of the analysis to two representative wings.

INTRODUCTION

Interest in the characteristics of the laminar boundary layer has increased in recent years because, under certain conditions, the boundary layer may remain laminar over large areas of airplanes and missiles. For example, Van Driest (ref. 1) has shown theoretically that if the solid boundary is cooled sufficiently, the laminar boundary layer can be stabilized regardless of Reynolds number at Mach numbers between 1 and 9. Sternberg (ref. 2) observed laminar boundary layers at Reynolds numbers as high as 50×10^6 in flight tests of the V-2 rocket. Laminar boundary layers may also be expected in flight at very high altitude where the density, and hence the Reynolds number per unit length, will be low.

Solutions of the compressible laminar-boundary-layer equations for the special case of zero pressure gradient have been obtained by several authors. The theory of Chapman and Rubesin (ref. 3), for example, presents a very simple method for calculating boundary-layer characteristics over a flat plate with arbitrary heat transfer. The more recent, and in general more exact, studies of Klunker and McLean (refs. 4 and 5), Van Driest (ref. 6), Young and Janssen (ref. 7), and Moore (ref. 8) have demonstrated that the theory of Chapman and Rubesin yields excellent results for reasonably low ambient air temperatures at Mach numbers up to about 5.

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Solutions for the more general case of arbitrary heat transfer and arbitrary pressure gradient are still in an early stage of development. Tani, in a little known paper (ref. 9), used a perturbation procedure to obtain direct solutions of the boundary-layer differential equations with a Falkner-Skan type external velocity distribution ($u_e \sim x^m$) and heat transfer. Results are easily obtainable from tabulated functions, but are limited to a Prandtl number of 1, small Mach numbers, and small rates of heat transfer. Furthermore, the Falkner-Skan type of external velocity distribution is not appropriate for supersonic flow over thin wings. Ginzel (ref. 10), Kalikhman (ref. 11), and Libby and Morduchow (extension of ref. 12) have obtained solutions of the compressible laminar-boundary-layer equations by an extended Pohlhausen method. However, the accuracy of the Pohlhausen method under conditions of heat transfer at high speeds has not been determined. In addition, the amount of work required in a particular application of references 10 and 11 is prohibitive because the simultaneous numerical solution of two differential equations is required. Libby and Morduchow avoid this difficulty by the additional assumption that certain variable quantities remain constant over the entire length of boundary-layer development.

The purpose of the present report is to present a method of solution developed at the NACA Lewis laboratory that is free of many of the limitations of references 9 to 12. An accurate method for calculating velocity and temperature profiles and skin-friction and heat-transfer characteristics for the compressible laminar boundary layer with heat transfer and a small pressure gradient is derived. The permissible pressure gradient may be of a form and magnitude usually encountered over thin aerodynamic shapes in supersonic flight. The solution is obtained by a method of perturbation on the flat-plate solution of Chapman and Rubesin; it constitutes the first two terms of a Maclaurin series expansion in terms of the free-stream velocity gradient parameter. The method involves the direct solution of the boundary-layer differential equations. Although the theory applies for any constant Prandtl number, tabulated results presented in this report apply, in general, for a Prandtl number of 0.72. For the case of heat transfer, results are limited to an isothermal wall.

Solutions of the first-order perturbation equations are presented in tabular form, so that the amount of effort required in a particular application is reduced to the arithmetic combination of several tabulated values. A section of the report entitled "APPLICATION OF ANALYSIS" is included in order to facilitate the application of results in practical applications.

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ASSUMPTIONS AND LIMITATIONS

The following simplifying assumptions and limitations are imposed in addition to the usual boundary-layer assumptions:

(1) The ratio of the velocity at the outer edge of the boundary layer u_e to a reference velocity u_r can be represented by

$$\frac{u_e}{u_r} = 1 + \varepsilon \; a_N x^N \tag{1}$$

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where the repeated index N indicates a summation over several values of N. (All symbols used in this report are defined in appendix A.) In equation (1) the quantity $a_N x^N$ represents the shape of the deviation of u_e from a constant value, while ε represents the magnitude of this deviation. The quantity ε is assumed small as compared with unity, whereas the quantity $a_N x^N$ is of normal order of magnitude. This type of external velocity distribution is capable of representing in form and magnitude those encountered over thin aerodynamic shapes at Mach numbers greater than 1.

(2) The temperature of the solid boundary is constant under conditions of heat transfer. (Under conditions of zero heat transfer the wall temperature will be a calculated function of the pressure distribution.)

(3) The viscosity and temperature are related linearly by the following expression:

$$\frac{\mu}{\mu_r} = C \frac{t}{t_r} \tag{2}$$

Chapman and Rubesin have shown that solutions of the boundary-layer equations based on equation (2) agree well with more exact solutions for flat-plate flows at Mach numbers less than 5 if the constant C is determined by matching equation (2) with Sutherland's relation at the solid boundary

$$C = \sqrt{\frac{t_w}{t_r}} \frac{(t_r + S)}{(t_w + S)} \tag{3}$$

This assumption should also be reasonable for flows with slight streamwise pressure gradients, especially when the wall temperature is constant. For a nonisothermal wall an average wall temperature should be used in equation (3), as suggested in reference 3.

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(4) The Prandtl number and specific heat are constant throughout the boundary layer. The restriction imposed by this assumption is not great because both Pr and c_p vary only slightly at moderate temperatures. A Prandtl number of 0.72 was used in all calculations.

GOVERNING EQUATIONS

Differential equations and boundary conditions. - The equations governing the steady laminar flow of a viscous compressible fluid in a thin boundary layer are the momentum equations

$$uu_x + vu_y = -\frac{1}{\rho} p_x + \frac{1}{\rho} (\mu u_y)_y \tag{4a}$$

$$p_y = 0 \tag{4b}$$

the equation of continuity

$$(\rho u)_x + (\rho v)_y = 0 \tag{5}$$

the energy equation

$$\rho\, c_p(ut_x + vt_y) = up_x + (kt_y)_y + \mu(u_y)^2 \tag{6}$$

and the equation of state

$$p = \rho Rt \tag{7}$$

The following boundary conditions are imposed on the momentum and energy equations:

$$
\left.
\begin{aligned}
&u(x,0) = 0 && u(x,\infty) = u_e \\
&v(x,0) = 0 && t(x,\infty) = t_e \\
&t(x,0) = t_w \text{ (heat transfer)} && \\
&t_y(x,0) = 0 \text{ (zero heat transfer)} &&
\end{aligned}
\right\} \tag{8}
$$

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At the outer edge of the boundary layer, velocity and pressure are related by the Bernoulli equation, which is

$$
\frac{dp_e}{dx} = -\rho_e u_e \frac{du_e}{dx} \tag{9}
$$

The energy equation that applies at the outer edge of the boundary layer is

$$
c_p T = c_p t_e + \frac{u_e^2}{2} \tag{10}
$$

<u>Transformation of Howarth</u>. - In reference 13 Howarth introduced a transformation which, when applied to the momentum and energy equations, yields equations similar in form to the incompressible-boundary-layer equations. First, it is convenient to introduce the dimensionless variables

$$
\left.
\begin{aligned}
&p^* = p/p_r && u^* = u/u_r \\
&t^* = t/t_r && v^* = v/u_r \\
&\rho^* = \rho/\rho_r && \mu^* = \mu/\mu_r
\end{aligned}
\right\} \tag{11}
$$

Howarth's transformation proceeds as follows: The independent variables x and y are related to the variables x and n according to the following transformation:

$$
\left.
\begin{aligned}
&x \equiv x \\
&n \equiv \sqrt{\frac{p^*}{C}} \int_0^y \frac{1}{t^*} dy
\end{aligned}
\right\} \tag{12}
$$

where n distorts the scale in the direction normal to the surface. The derivatives with respect to x and y can be expressed as

$$
\left.
\begin{aligned}
\left.\frac{\partial}{\partial x}\right)_y &= \left.\frac{\partial}{\partial x}\right)_n + \frac{\partial n}{\partial x}\frac{\partial}{\partial n} \\
\frac{\partial}{\partial y} &= \sqrt{\frac{p^*}{C}}\,\frac{1}{t^*}\,\frac{\partial}{\partial n}
\end{aligned}
\right\} \tag{13}
$$

Equation (5) is satisfied by a stream function defined as follows: 2968

$$
\rho^* u^* = \psi_y
$$

$$
\rho^* v^* = -\psi_x
$$

The stream function $\psi(x,y)$ can be related to a transformed function $\varphi(x,n)$ by

$$
\psi(x,y) \equiv \sqrt{Cp^*}\,\varphi(x,n) \tag{14}
$$

The velocity components now become

$$
\left.
\begin{aligned}
u &= u_r \varphi_n \\
v &= -\frac{u_r\sqrt{Cp^*}}{\rho^*}\left(\varphi_x + \varphi_n n_x + \frac{1}{2}\frac{p_x^*}{p^*}\varphi\right)
\end{aligned}
\right\} \tag{15}
$$

Substitution of equations (2), (7), (9), and (12) to (15) into equation (4a) yields the momentum equation in the transformed (x,n) plane

$$
\varphi_n \varphi_{nx} - \varphi_x \varphi_{nn} - \frac{\nu_r}{u_r}\varphi_{nnn} = \frac{u_e^*}{t_e^*}\frac{du_e^*}{dx}\left[t^* - \frac{1}{2}\gamma M_r^2 \varphi\,\varphi_{nn}\right] \tag{16}
$$

The energy equation in the x,n-plane becomes

$$
\begin{aligned}
\varphi_n t_x^* - \varphi_x t_n^* - \frac{\nu_r}{u_r\,Pr}t_{nn}^* = (\gamma-1)\,M_r^2\frac{\nu_r}{u_r}\varphi_{nn}^2 - \\
M_r^2\frac{u_e^*}{t_e^*}\frac{du_e^*}{dx}\left[\frac{\gamma}{2}\varphi\,t_n^* + (\gamma-1)\,t^*\varphi_n\right]
\end{aligned} \tag{17}
$$

The following boundary conditions apply to equations (16) and (17):

$$\varphi(x,0) = 0 \qquad \varphi_n(x,\infty) = u_e^*$$

$$\varphi_n(x,0) = 0$$

$$t^*(x,0) = t_w^* \text{ (heat transfer)}$$

$$t_n^*(x,0) = 0 \text{ (zero heat transfer)}$$

$$t^*(x,\infty) = t_e^*$$

PERTURBATION ANALYSIS

Expansion of momentum and energy equations in powers of ε. - For the special case of $u_e^* = 1$ (zero pressure gradient), equations (16) and (17) become identical to the momentum and energy equations solved by Chapman and Rubesin. It therefore appears logical to let u_e^* differ from unity by a small amount in order to obtain a perturbation solution for flows with small pressure gradients. As discussed under ASSUMPTIONS AND LIMITATIONS, the external velocity at the outer edge of the boundary layer is taken to be of the following form:

$$u_e^* = 1 + \varepsilon\, a_N x^N \tag{1}$$

Substitution of equations (1) and (7) into equations (9) and (10) and elimination of higher-order terms yield:

$$p^* = 1 - \gamma M_r^2\, \varepsilon\, a_N x^N \tag{18}$$

and

$$t_e^* = 1 - (\gamma-1) M_r^2\, \varepsilon\, a_N x^N \tag{19}$$

Within the boundary layer the stream function and temperature are replaced by their Maclaurin series expansions in terms of the velocity gradient parameter ε:

$$\varphi(x,n,\varepsilon) = \overline{\varphi}(x,n) + \varepsilon\, a_N\, \overline{\overline{\varphi}}_N(x,n) + \varepsilon^2\, a_{NM}\, \overline{\overline{\overline{\varphi}}}_{NM}(x,n) + \ldots \tag{20}$$

$$t^*(x,n,\varepsilon) = \overline{t}(x,n) + \varepsilon\, a_N\, \overline{\overline{t}}_N(x,n) + \varepsilon^2\, a_{NM}\, \overline{\overline{\overline{t}}}_{NM}(x,n) + \ldots \tag{21}$$

A sequence of momentum and energy equations is obtained by substitution of equations (1) and (18) to (21) into equations (16) and (17), and by equating coefficients of like powers of ε. The zero-order equations, obtained by equating coefficients of $(\varepsilon)^0$, are:

$$\bar{\varphi}_n \, \bar{\varphi}_{nx} - \bar{\varphi}_x \, \bar{\varphi}_{nn} - \frac{\nu_r}{u_r} \bar{\varphi}_{nnn} = 0 \tag{22}$$

$$\bar{\varphi}(x,0) = \bar{\varphi}_n(x,0) = 0 \qquad \bar{\varphi}_n(x,\infty) = 1$$

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and

$$\bar{\varphi}_n \, \bar{t}_x - \bar{\varphi}_x \, \bar{t}_n - \frac{\nu_r}{u_r \, Pr} \bar{t}_{nn} = (\gamma-1) \frac{\nu_r \, M_r^2}{u_r} (\bar{\varphi}_{nn})^2 \tag{23}$$

$$\bar{t}(x,0) = t_w^* \text{ (heat transfer)}$$

$$\bar{t}_n(x,0) = 0 \text{ (zero heat transfer)}$$

$$\bar{t}(x,\infty) = 1$$

Equating coefficients of ε yields the first-order equations:

$$\bar{\varphi}_n \, \bar{\bar{\varphi}}_{Nnx} + \bar{\varphi}_{nx} \, \bar{\bar{\varphi}}_{Nn} - \bar{\varphi}_x \, \bar{\bar{\varphi}}_{Nnn} - \bar{\varphi}_{nn} \, \bar{\bar{\varphi}}_{Nx} - \frac{\nu_r}{u_r} \bar{\bar{\varphi}}_{Nnnn}$$

$$= N \, x^{N-1} \left(\bar{t} - \frac{\gamma}{2} M_r^2 \, \bar{\varphi} \, \bar{\varphi}_{nn}\right) \tag{24}$$

$$\bar{\bar{\varphi}}_N(x,0) = \bar{\bar{\varphi}}_{Nn}(x,0) = 0 \qquad \bar{\bar{\varphi}}_{Nn}(x,\infty) = x^N$$

and

$$\bar{\bar{\varphi}}_{Nn} \, \bar{t}_x + \bar{\varphi}_n \, \bar{\bar{t}}_{Nx} - \bar{\bar{\varphi}}_{Nx} \, \bar{t}_n - \bar{\varphi}_x \, \bar{\bar{t}}_{Nn} - \frac{\nu_r}{u_r \, Pr} \bar{\bar{t}}_{Nnn}$$

$$= 2 \frac{\nu_r}{u_r} (\gamma-1) \, M_r^2 \, \bar{\varphi}_{nn} \, \bar{\bar{\varphi}}_{Nnn} - N x^{N-1} \, M_r^2 \left[\frac{\gamma}{2} \bar{\varphi} \, \bar{t}_n + (\gamma-1) \, \bar{t} \, \bar{\varphi}_n\right] \tag{25}$$

$$\bar{\bar{t}}_N(x,0) = 0 \quad \text{or} \quad \bar{\bar{t}}_{Nn}(x,0) = 0 \qquad \bar{\bar{t}}_N(x,\infty) = -\,(\gamma-1) x^N \, M_r^2$$

The higher-order equations are obtained by equating coefficients of ε^2, ε^3, and so forth. If it is assumed that the functions $\bar{\varphi}$, $\bar{\bar{\varphi}}$, $\bar{\bar{\bar{\varphi}}}$, and so forth, and the functions $\bar{t}$, $\bar{\bar{t}}$, $\bar{\bar{\bar{t}}}$, and so forth, are of the same order of magnitude, then, since ε is postulated to be small, all additional contributions will be of second or higher order and hence may be neglected in a first-order treatment. Further justification for neglecting the higher-order equations comes from reference 14, where it is shown that for incompressible flow the function $\bar{\bar{\bar{\varphi}}}$ is numerically much smaller than $\bar{\varphi}$ and $\bar{\bar{\varphi}}$. The second-order terms therefore should contribute very little to the solution of the boundary-layer equations for flows with small pressure gradients.

The permissible magnitude of the pressure gradient depends largely on the length of run over which it acts. For example, a very small pressure gradient can cause laminar separation if it acts over a large distance. The present method can be applied only if all the deviations in boundary-layer characteristics caused by the pressure gradient are small.

The only dependent variable appearing in equation (22) is $\bar{\varphi}$, so that the solution of this equation is independent of all following equations. Furthermore, each succeeding equation involves only one new dependent variable, so that each equation can be solved in principle once the preceding equations have been solved. Equations (22) to (25) still contain two independent variables, however, and require further reduction to make them amenable to solution.

<u>Solution of zero-order equation</u>. - The zero-order equations may be transformed to ordinary differential equations by introduction of the Blasius characteristic variable η:

$$\eta \equiv \frac{n}{2} \sqrt{\frac{u_r}{\nu_r\, x}} \tag{26}$$

The stream function $\bar{\varphi}(x,n)$ is related to a new function $f(\eta)$ as follows:

$$\bar{\varphi}(x,n) = \sqrt{\frac{\nu_r\, x}{u_r}}\, f(\eta) \tag{27}$$

The temperature in the x,n-coordinate system is equal to the temperature in the η-system

$$\bar{t}(x,n) = \bar{t}(\eta) \tag{28}$$

With the aid of equations (26), (27), and (28), equations (22) and (23) can be written

$$f''' + ff'' = 0 \tag{29}$$

$$\bar{t}'' + \Pr f\,\bar{t}' = -\,(\Pr)\left(\frac{\gamma-1}{4}\right) M_r^2\,(f'')^2 \tag{30}$$

where the primes indicate differentiation with respect to η. The boundary conditions of f and t are

$$\begin{aligned} &f(0) = 0 \qquad f'(\infty) = 2 \\ &f'(0) = 0 \\ &\bar{t}(0) = t_w^* \quad \text{or} \quad \bar{t}'(0) = 0 \\ &\bar{t}(\infty) = 1 \end{aligned}$$

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Equation (29) is the well-known Blasius equation which has been solved by several investigators. In order to eliminate M_r^2 as a parameter in the solution of equation (30), this equation is split into two parts in the following manner:

$$\bar{t}(\eta) = 1 + \frac{\gamma-1}{2} M_r^2\, r(\eta) + K\, s(\eta) \tag{31}$$

where $r(\eta)$ and $s(\eta)$ satisfy the following equations:

$$r'' + \Pr f\, r' = -\frac{\Pr}{2}\,(f'')^2 \tag{32}$$

$$s'' + \Pr f\, s' = 0 \tag{33}$$

The following boundary conditions are applied to equations (32) and (33):

$$\begin{aligned} &r'(0) = 0 && r(\infty) = 0 \\ &s'(0) = -\,[f''(0)]^{\Pr} && s(\infty) = 0 \end{aligned}$$

Although the solution of equations (32) and (33) can be written in terms of quadratures, as shown in reference 3, the numerical solution of the differential equations was found more convenient. Numerical solutions, as discussed in appendix B, were made by Lynn U. Albers.

The functions $f''(0)$, $r(0)$, $s(0)$, and $s'(0)$ are listed in table I; all other functions resulting from the solution of the zero-order equations can be found in table II. The constant K (eq. (31)) is related to the rate of heat transfer and hence to the wall temperature. Its value is determined by solving equation (31) at $\eta = 0$.

$$K = \frac{1}{s(0)} \left[(\bar{t}_w - 1) - \frac{\gamma-1}{2} M_r^2 \, r(0) \right] \tag{34}$$

where the wall temperature ratio $\bar{t}_w = \frac{t_w}{t_r}$ is, in general, prescribed. For the case of zero heat transfer, K vanishes.

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<u>Solution of first-order equations</u>. - The first-order momentum and energy equations can also be transformed to ordinary differential equations with the aid of the Blasius variable together with the following functions:

$$\bar{\bar{\varphi}}_N(x,n) = 2 \sqrt{\frac{\nu_r \, x}{u_r}} \, x^N \, g_N(\eta) \tag{35}$$

and

$$\bar{\bar{t}}_N(x,n) = - (\gamma-1) \, M_r^2 \, x^N \, h_N(\eta) \quad \text{(zero heat transfer)} \tag{36}$$

$$\bar{\bar{t}}_N(x,n) = - (\gamma-1) \, M_r^2 \, x^N \, H_N(\eta) \quad \text{(heat transfer)} \tag{37}$$

Equations (24) and (25) are now written

$$\begin{aligned} & g_N''' + f \, g_N'' - 2N \, f' \, g_N' + (2N + 1) \, f'' \, g_N \\ & = - 4N \left\{ 1 + M_r^2 \left[\left(\frac{\gamma-1}{2} \right) r - \frac{\gamma}{8} \, f f'' \right] + Ks \right\} \end{aligned} \tag{38}$$

and

$$\begin{aligned} & h_N'' + Pr \, f h_N' - 2Pr \, N f' h_N \\ & = Pr \left[\frac{4N + 2}{(\gamma-1) M_r^2} \, g_N \, \bar{t}' + f'' \, g_N'' - \frac{\gamma}{\gamma-1} \, N \, f \bar{t}' - 2N f' \bar{t} \right] \end{aligned} \tag{39}$$

The function $H_N(\eta)$ satisfies the same equation as is satisfied by $h_N(\eta)$ (eq. (41)), but is subject to different boundary conditions. The boundary conditions are

$$g_N(0) = 0 \qquad g_N'(\infty) = 1$$

$$g_N'(0) = 0$$

$$h_N'(0) = 0 \qquad h_N(\infty) = 1 \quad \text{(zero heat transfer)}$$

$$H_N(0) = 0 \qquad H_N(\infty) = 1 \quad \text{(heat transfer)}$$

The solution of equation (38) can be obtained in closed form for the special cases of $N = -\frac{1}{2}$ (ref. 15) and $N = 0$ (see appendix C). In the general case, however, the equation was solved numerically. In order to obtain a numerical solution which applies over a range of Mach numbers and heat-transfer rates, the parameters M_r and K were eliminated from equation (38) by splitting the function g into a linear combination of three independent functions:

$$g_N = g_{N1} + M_r^2\, g_{N2} + K\, g_{N3} \tag{40}$$

where the three new functions satisfy the following equations:

$$g_{N1}''' + fg_{N1}'' - 2Nf'g_{N1}' + (2N + 1)\, f''g_{N1} = -\, 4N \tag{41}$$

$$g_{N2}''' + fg_{N2}'' - 2Nf'g_{N2}' + (2N + 1)\, f''g_{N2} = \frac{N}{2}\, [\gamma ff'' - 4(\gamma-1)r] \tag{42}$$

$$g_{N3}''' + fg_{N3}'' - 2Nf'g_{N3}' + (2N + 1)\, f''g_{N3} = -\, 4Ns \tag{43}$$

$$g_{Ni}(0) = g_{Ni}'(0) = 0 \qquad (i = 1,2,3)$$

$$g_{N1}'(\infty) = 1 \qquad g_{N2}'(\infty) = g_{N3}'(\infty) = 0$$

The first-order energy equation can be solved in closed form for $N = -\frac{1}{2}$ (ref. 15) and $N = 0$ (appendix C). For zero heat transfer, a closed form solution of equation (39) for $Pr = 1$ can also be obtained for all values of N, as shown in appendix D. For other cases the solution of equation (39) is again found numerically after several parameters have been eliminated by replacing the equation by the following system:

$$h_N = h_{N1} + M_r^2\, h_{N2} \tag{44}$$

where

$$h_{N1}'' + Pr\ fh_{N1}' - 2Pr\ Nf'h_{N1} = Pr\left[(2N + 1)g_{N1}r' + f''g_{N1}'' - 2Nf'\right] \tag{45}$$

$$h_{N2}'' + Pr\ fh_{N2}' - 2Pr\ Nf'h_{N2}$$

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$$= Pr\left[(2N + 1)g_{N2}\ r' + f''g_{N2}'' - \frac{\gamma\ Nr'f}{2} - N(\gamma-1)rf'\right] \tag{46}$$

$$h_{Ni}'(0) = 0 \qquad h_{N1}(\infty) = 1 \qquad h_{N2}(\infty) = 0 \qquad (i = 1,2)$$

Equations (44) to (46) apply for the case of zero heat transfer. For flows with heat transfer, the following equations are obtained:

$$H_N = H_{N1} + M_r^2\ H_{N2} + KH_{N3} + \frac{K}{M_r^2}\ H_{N4} + \frac{K^2}{M_r^2}\ H_{N5} \tag{47}$$

The functions $H_{N1}(\eta)$ and $H_{N2}(\eta)$ satisfy the same equations or are satisfied by $h_{N1}(\eta)$ and $h_{N2}(\eta)$, but are subject to modified boundary conditions. The remaining functions in equation (47) satisfy the following equations:

$$H_{N3}'' + Pr\ fH_{N3}' - 2Pr\ Nf'H_{N3} = Pr\left[(2N + 1)\left(\frac{2g_{N2}s'}{\gamma-1} + r'\ g_{N3}\right) + f''g_{N3}'' - \frac{\gamma Nfs'}{\gamma-1} - 2Nf's\right] \tag{48}$$

$$H_{N4}'' + Pr\ fH_{N4}' - 2Pr\ Nf'H_{N4} = Pr\left[(2N + 1)\left(\frac{2g_{N1}s'}{\gamma-1}\right)\right] \tag{49}$$

$$H_{N5}'' + Pr\ fH_{N5}' - 2Pr\ Nf'H_{N5} = Pr\left[(2N + 1)\left(\frac{2g_{N3}s'}{\gamma-1}\right)\right] \tag{50}$$

$$H_{Ni}(0) = 0 \qquad H_{N1}(\infty) = 1 \qquad (i = 1,2,3,4,5)$$

$$H_{N2}(\infty) = H_{N3}(\infty) = H_{N4}(\infty) = H_{N5}(\infty) = 0$$

The elimination of M_r and K as parameters in the first-order equations has thus yielded ten equations for each value of N. These equations were solved numerically for $N = 1, 2,$ and 3, as discussed in appendix B. The functions g_N', which represent the first-order velocity corrections, and h_N and H_N, which represent the first-order temperature corrections, are presented graphically in figures 1, 2, and 3. Tabulated results of all solutions of the first-order equations can be found in tables III, IV, and V. Initial values are also tabulated in table I.

The solutions of the zero- and first-order equations can now be combined to yield velocity and temperature profiles, skin-friction and heat-transfer coefficients, recovery factors, and displacement thickness. 2968

BOUNDARY-LAYER CHARACTERISTICS

Velocity and temperature profiles. - The dimensionless velocity u^* is related to the characteristic variable η through equations (15), (20), (27), and (35) in the following manner:

$$u^* \cong \frac{1}{2} f'(\eta) + \varepsilon\, a_N x^N g_N'(\eta) \tag{51}$$

From equations (21), (28), (31), and (36) or (37), the temperature profile can be expressed in terms of η as

$$t^* \cong 1 + \frac{\gamma-1}{2} M_r^2 \left[r(\eta) - 2\varepsilon\, a_N x^N h_N(\eta)\right] \tag{52}$$

for the case of zero heat transfer. For flows with arbitrary heat transfer, the following expression applies:

$$t^* \cong 1 + K\, s(\eta) + \frac{\gamma-1}{2} M_r^2 \left[r(\eta) - 2\varepsilon\, a_N x^N H_N(\eta)\right] \tag{53}$$

Equations (51), (52), and (53) represent the velocity and temperature profiles as functions of η. The transformation to the physical (x,y-) plane, according to equations (12) and (26), is

$$y = 2\sqrt{\frac{\nu_r x C}{p^*\, u_r}} \int_0^\eta t^*\, d\eta \tag{54}$$

The value of t^* can be obtained from equations (52) or (53), while p^* is given by equation (18). Thus, for zero heat transfer,

$$y \cong 2\sqrt{\frac{\nu_r\ xC}{u_r}}\left\{\left[1 + \frac{\gamma}{2} M_r^2\ \varepsilon\ a_N x^N\right]\left[\eta + \frac{\gamma-1}{2} M_r^2\ Ir(\eta)\right] - (\gamma-1)M_r^2\ \varepsilon\ a_N x^N Ih_N\right\} \tag{55}$$

and for arbitrary rates of heat transfer,

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$$y \cong 2\sqrt{\frac{\nu_r\ xC}{u_r}}\left\{\left[1 + \frac{\gamma}{2} M_r^2\ \varepsilon\ a_N x^N\right]\left[\eta + \frac{\gamma-1}{2} M_r^2\ Ir(\eta) + KI\ s(\eta)\right] - (\gamma-1)\,\varepsilon\ a_N x^N\ M_r^2\ IH_N(\eta)\right\} \tag{56}$$

Skin friction and heat transfer. - The shearing stress at a point in the boundary layer can be obtained from equations (2), (13), and (15):

$$\tau \equiv \mu\ \frac{\partial u}{\partial y} = \mu_r\ u_r\sqrt{Cp^*}\ \varphi_{nn} \tag{57}$$

In terms of the Blasius variable η and after substitution of equation (18) for p^*, equation (57) becomes

$$\tau \cong \frac{u_r}{4}\sqrt{\frac{\mu_r\ \rho_r\ u_r\ C}{x}}\left\{f''(\eta) + 2\ \varepsilon\ a_N x^N\left[g_N''(\eta) - \frac{\gamma}{4} M_r^2\ f''(\eta)\right]\right\} \tag{58}$$

Local and average skin-friction coefficients are obtained from the wall shearing stress and the following respective definitions:

$$c_f = \frac{\tau_w}{\frac{1}{2}\ \rho_r\ u_r^2} \tag{59}$$

and

$$C_F = \frac{1}{\frac{1}{2}\ \rho_r\ u_r^2\ x}\int_0^x \tau_w\ dx \tag{60}$$

A local skin-friction parameter is obtained from equations (58) and (59):

$$c_f\sqrt{\frac{Re}{C}} \cong \frac{1}{2}\left\{f''(0) + 2\varepsilon\ a_N x^N\left[g_N''(0) - \frac{\gamma}{4} M_r^2\ f''(0)\right]\right\} \tag{61}$$

The average friction drag parameter, obtained from equations (58) and (60), is

$$C_F\sqrt{\frac{Re}{C}} \cong f''(0) + \frac{2\varepsilon\ a_N x^N}{2N + 1}\left[g_N''(0) - \frac{\gamma}{4} M_r^2\ f''(0)\right] \tag{62}$$

The local rate of heat transfer from the surface is given by

$$q \equiv -\ k\left.\frac{\partial t}{\partial y}\right)_w = -\ \frac{c_p\ t_r\ \mu_r}{Pr}\sqrt{Cp^*}\left.t_n^*\right)_w$$

$$\cong -\ \frac{c_p}{2\ Pr}\ t_r\sqrt{\frac{\mu_r\ \rho_r\ u_r\ C}{x}}\Bigg\{K\ s'(0)\ -$$

$$(\gamma-1)\ M_r^2\ \varepsilon\ a_N x^N\left[H_N'(0) + \frac{\gamma}{2(\gamma-1)}\ K\ s'(0)\right]\Bigg\} \tag{63}$$

A dimensionless heat-transfer parameter can now be written as follows:

$$\frac{Nu}{\sqrt{C\ Re}} \cong \frac{1}{2(t_{aw}^* - t_w^*)}\Bigg\{K\ s'(0)\ -$$

$$(\gamma-1)\ M_r^2\ \varepsilon\ a_N x^N\left[H_N'(0) + \frac{\gamma}{2(\gamma-1)}\ K\ s'(0)\right]\Bigg\} \tag{64}$$

where the dimensionless adiabatic wall temperature is

$$t_{aw}^* \cong 1 + \frac{\gamma-1}{2} M_r^2\left[r(0) - 2\varepsilon\ a_N x^N\ h_N(0)\right] \tag{65}$$

Temperature recovery factor. - The temperature recovery factor is derived from equations (19) and (65):

$$F_R \equiv \frac{t_{aw}^* - t_e^*}{T^* - t_e^*} \cong r(0) + 2\varepsilon\ a_N x^N\left[1 - h_N(0) - r(0)\right] \tag{66}$$

It is evident from the computed results that $h_{N1}(0)$ varies very little with N and is approximately equal to $1 - r(0)$. With the aid of equation (44), equation (66) is therefore reduced to the following:

$$F_R \cong r(0) - 2\varepsilon\ a_N x^N\ M_r^2\ h_{N2}(0) \tag{67}$$

Displacement thickness. - The boundary-layer displacement thickness is, by definition,

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$$\delta^* \equiv \int_0^\infty \left(1 - \frac{\rho u}{\rho_e\, u_e}\right) dy \tag{68}$$

$$= 2\sqrt{\frac{C\,\nu_r\,x}{p^*\,u_r}} \int_0^\infty \left(t^* - \frac{t_e^*}{u_e^*}\,u^*\right) d\eta \tag{69}$$

With the appropriate expressions for p^*, t^*, and u^*, equation (69) becomes, for flows with heat transfer,

$$\delta^* \cong 2\sqrt{\frac{C\,\nu_r\,x}{u_r}} \int_0^\infty \left\{\left[1 + \frac{\gamma}{2} M_r^2\, \varepsilon\, a_N x^N\right]\left[\left(1 - \frac{1}{2} f'\right) + Ks + \frac{\gamma-1}{2} M_r^2\, r\right] + \right.$$

$$\left. \varepsilon\, a_N x^N \left[\frac{1}{2} f' - g_N'\right] + \frac{\gamma-1}{2} M_r^2\, \varepsilon\, a_N x^N \left[f' - 2H_N\right]\right\} d\eta \tag{70}$$

Integration of equation (70) yields, for $Pr = 0.72$,

$$\delta^* \cong \sqrt{\frac{C\,\nu_r\,x}{u_r}} \left\{\left[1 + \frac{\gamma}{2} M_r^2\, \varepsilon\, a_N x^N\right]\left[1.7208 + 4.0218\,K + \right.\right.$$

$$\left.\left. 1.1094\,(\gamma-1)\,M_r^2\right] + \varepsilon\, a_N x^N \left[\alpha_N + (\gamma-1)\,M_r^2\, B_N\right]\right\} \tag{71}$$

For the case of zero heat transfer K will vanish and B_N is replaced by β_N. The relation between the functions appearing in equation (71) and the functions tabulated in table VI is:

$$\alpha_N = \alpha_{N1} + M_r^2\, \alpha_{N2} + K\, \alpha_{N3}$$

$$\beta_N = \beta_{N1} + M_r^2\, \beta_{N2}$$

$$B_N = B_{N1} + M_r^2\, B_{N2} + K\, B_{N3} + \frac{K}{M_r^2}\, B_{N4} + \frac{K^2}{M_r^2}\, B_{N5}$$

APPLICATION OF ANALYSIS

Before the results of the previous section may be applied it is necessary to determine the quantity ε, the coefficients a_N, and the reference conditions. The quantities ε and a_N, which represent the magnitude and form, respectively, of the external velocity distribution, are determined from potential-flow theory or from experimental measurements. Because the results of this report apply primarily to the flow over thin two-dimensional wings at Mach numbers greater than 1, ε and a_N as obtained by linearized theory (ref. 16) will be presented herein. It is assumed that the coordinates of a wing section are known and can be fitted by a polynomial of fourth or lesser degree:

$$Y = b_1x + b_2x^2 + b_3x^3 + b_4x^4 \tag{72}$$

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The values of ε, a_N, and u_r are obtained by matching the expression for the velocity distribution obtained from linearized theory with equation (1):

$$\frac{u_e}{u_r} = 1 + \varepsilon(a_1x + a_2x^2 + a_3x^3) \tag{1}$$

where

$$\left.\begin{aligned} u_r &= \left(1 - \frac{a_1}{\sqrt{M_\infty^2 - 1}}\right)u_\infty \\ \varepsilon &= \frac{-2b_2}{\sqrt{M_\infty^2 - 1}} \\ a_1 = 1 \qquad a_2 &= \frac{3b_3}{2b_2} \qquad a_3 = \frac{2b_4}{b_2} \end{aligned}\right\} \tag{73}$$

If the velocity distribution over the wing were known experimentally, the starting point of the calculation would be equation (1), with u_r, ε, and a_N determined by fitting a polynomial to the measured velocities. If, in a particular application, a_2 or a_3 is much smaller than 1, then that term need not be included in the solution.

The reference Mach number and temperature are obtained from equations (10) and (73):

$$M_r^2 = \frac{M_\infty^2\left(\frac{u_r}{u_\infty}\right)^2}{1 + \frac{\gamma-1}{2}M_\infty^2\left[1 - \left(\frac{u_r}{u_\infty}\right)^2\right]} \tag{74}$$

$$t_r = t_\infty \sqrt{\frac{1 + \frac{\gamma-1}{2} M_\infty^2}{1 + \frac{\gamma-1}{2} M_r^2}} \tag{75}$$

The results of the analysis will now be summarized as they are needed in a particular application. In the following equations the functions $g_n(\eta)$, $h_n(\eta)$, and $H_n(\eta)$ frequently appear. They are related to the tabulated functions in the following manner:

$$g_N(\eta) = g_{N1}(\eta) + M_r^2\, g_{N2}(\eta) + K g_{N3}(\eta) \tag{40}$$

$$h_N(\eta) = h_{N1}(\eta) + M_r^2\, h_{N2}(\eta) \tag{44}$$

$$H_N(\eta) = H_{N1}(\eta) + M_r^2\, H_{N2}(\eta) + K H_{N3}(\eta) + \frac{K}{M_r^2} H_{N4}(\eta) + \frac{K^2}{M_r^2} H_{N5}(\eta) \tag{47}$$

The constant K is related to the given wall temperature for flows with arbitrary rates of heat transfer:

$$K = \frac{1}{s(0)} \left\{ \frac{t_w}{t_r} - 1 - \frac{\gamma-1}{2} M_r^2\, r(0) \right\} \tag{34}$$

where $s(0)$ and $r(0)$ appear in table I. (For flows with zero heat transfer, $K = 0$.)

The velocity profile is given by

$$\frac{u}{u_r} = \frac{1}{2} f'(\eta) + \varepsilon\, a_N x^N\, g_N'(\eta) \tag{51}$$

where the repeated index N indicates a summation over all values of N. The temperature profile, for zero heat transfer, is

$$\frac{t}{t_r} = 1 + \frac{\gamma-1}{2} M_r^2 \left[r(\eta) - 2\varepsilon\, a_N x^N\, h_N(\eta) \right] \tag{52}$$

and for flows with heat transfer, is

$$\frac{t}{t_r} = 1 + K s(\eta) + \frac{\gamma-1}{2} M_r^2 \left[r(\eta) - 2\varepsilon\, a_N x^N\, H_N(\eta) \right] \tag{53}$$

These profiles can be obtained in terms of the physical variable y by the following relations between η and y: (a) for zero heat transfer,

$$y = 2\sqrt{\frac{\nu_r\, xC}{u_r}}\left\{\left[1 + \frac{\gamma}{2} M_r^2\, \varepsilon\, a_N x^N\right]\left[\eta + \frac{\gamma-1}{2} M_r^2\, Ir(\eta)\right] - (\gamma-1)\, M_r^2\, \varepsilon\, a_N x^N\, Ih_N(\eta)\right\} \tag{55}$$

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(b) for arbitrary rates of heat transfer,

$$y = 2\sqrt{\frac{\nu_r\, xC}{u_r}}\left\{\left[1 + \frac{\gamma}{2} M_r^2\, \varepsilon\, a_N x^N\right]\left[\eta + \frac{\gamma-1}{2} M_r^2\, Ir(\eta) + KIs(\eta)\right] - (\gamma-1)\, M_r^2\, \varepsilon\, a_N x^N\, IH_N(\eta)\right\} \tag{56}$$

The functions f, r, and s appear in table II. The functions g appear in table III; h, in table IV; and H, in table V.

The constant C is defined by

$$C = \sqrt{\frac{t_w}{t_r}\,\frac{(t_r + 216^\circ\ R)}{(t_w + 216^\circ\ R)}} \tag{3}$$

where t_w is given for flows with heat transfer, while for zero heat transfer a mean value of the adiabatic wall temperature is used. The adiabatic wall temperature is

$$t_{aw} = t_r\left\{1 + \frac{\gamma-1}{2} M_r^2 \left[r(0) - 2\varepsilon\, a_N x^N\, h_N(0)\right]\right\} \tag{65}$$

The temperature recovery factor is

$$F_R = r(0) - 2\varepsilon\, a_N x^N\, M_r^2\, h_{N2}(0) \tag{67}$$

The following results were found for local and average skin-friction coefficients and for a heat-transfer parameter:

$$c_f = \frac{1}{2}\sqrt{\frac{C}{Re}}\left\{f''(0) + 2\varepsilon\, a_N x^N \left[g_N''(0) - \frac{\gamma}{4} M_r^2\, f''(0)\right]\right\} \tag{61}$$

$$C_F = \sqrt{\frac{C}{Re}}\left\{f''(0) + \frac{2\varepsilon\ a_N x^N}{2N + 1}\left[g_N''(0) - \frac{\gamma}{4} M_r^2\ f''(0)\right]\right\} \tag{62}$$

and

$$Nu = \frac{\sqrt{C\ Re}}{2(t_{aw}^* - t_w^*)}\left\{Ks'(0) - (\gamma-1)\ M_r^2\ \varepsilon\ a_N x^N\left[H_N'(0) + \frac{\gamma}{2(\gamma-1)}\ Ks'(0)\right]\right\} \tag{64}$$

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where

$$Re \equiv \frac{u_r\ x}{\nu_r}$$

$$Nu \equiv \frac{C\ t_w^*\ x}{t_{aw}^* - t_w^*}\ t_y^*\Big)_w$$

All initial values [$f''(0)$, etc.] are tabulated in table I. The displacement thickness for flows with arbitrary heat-transfer rates is given by

$$\delta^* = \sqrt{\frac{\nu_r\ xC}{u_r}}\left\{\left[1 + \frac{\gamma}{2} M_r^2\ \varepsilon\ a_N x^N\right]\left[1.7208 + 4.0218\ K + (\gamma-1)(1.1094\ M_r^2)\right] + \varepsilon\ a_N x^N\left[\alpha_N + (\gamma-1)\ M_r^2\ B_N\right]\right\} \tag{71}$$

For flows with zero heat transfer, K will vanish and B_N is replaced by β_N in the last equation. (Values of α_N, β_N, and B_N can be found in table VI.)

The results of this analysis are not necessarily limited to the integral values of N for which calculations were made. Interpolation of the results presented in the tables will yield valid results for other values of N. Values for $N = 0$ are included in order to facilitate this interpolation (see appendix C and table I).

The equations presented in this section apply also for flat-plate flows. For this special case, $\varepsilon = 0$ and the reference conditions are equal to the undisturbed free-stream conditions.

DISCUSSION OF EXAMPLES

The results of the previous section were applied to two representative wings in order to determine the combined effects of heat transfer and pressure gradient on boundary-layer characteristics. Cross-sectional views of the forward portion of these wings are shown in figure 4. The first of the two wings has a constant adverse pressure gradient, while the second has a constant favorable pressure gradient. A maximum thickness ratio of 0.05 and a free-stream Mach number of 3 were chosen for the wing segments of both examples. The velocity and temperature distributions at the outer edge of the boundary layer are shown in figure 5.

The local skin-friction parameter $c_f\sqrt{Re/C}$ for both representative wings, computed by the present method, is presented in figure 6. The effect of pressure gradient in the absence of heat transfer $\left[\left(\frac{\partial t}{\partial y}\right)_w = 0\right]$ 2968
is to decrease skin friction for flows with adverse pressure gradients, and to increase skin friction for flows with favorable gradients. (For flows with zero pressure gradients, $c_f\sqrt{Re/C} = 0.664$ for all values of x as indicated by a dashed line in the figure.) The effects of pressure gradient are accentuated by adding heat to the boundary layer. For the present examples, the aforementioned decrease and increase in skin friction is doubled when the wall is heated to approximately four times the ambient air temperature. Sufficient cooling at the wall, on the other hand, appears to reverse the trend of the pressure gradient alone. Thus, for a wall temperature approximately equal to one-fourth the ambient air temperature, there is a slight increase in skin friction for flows with adverse pressure gradients; whereas there is a decrease in the case of favorable pressure gradients. The average friction drag parameter $C_F\sqrt{Re/C}$, as shown in figure 7, exhibits the same trends as the local skin friction.

The friction-drag parameter $C_F\sqrt{Re/C}$ is useful because it applies at all flight altitudes, and the actual velocity, density, and temperature need not be specified a priori. On the other hand, it is a misleading parameter, because the viscosity-temperature dependence factor C, which is a function of the wall temperature ratio, is affected by the rate of heat transfer at the wall. For this reason, the average friction drag coefficient multiplied by $\sqrt{Re}$ was found for the two representative wings at conditions existing at 35,000 feet, as shown in figure 8. The rate of change of friction drag along the surface is nearly the same as was shown in figure 7. The relative magnitudes of the friction drag curves are altered, however, so that the highest drag is found for the cooling case, while the lowest drag is obtained with the hot wall, regardless of the type of pressure gradient.

The heat-transfer parameter $Nu/\sqrt{Re\ C}$ for the two representative wings is plotted in figure 9. The local rate of heat transfer was found to increase along the wing when the pressure gradient and wall temperature were such that the skin friction decreased and vice versa.

The temperature recovery factor, as plotted in figure 10, was found to vary slightly as a result of the pressure gradient. The variation is

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of the same order of magnitude as the pressure gradient, and hence a much larger change might be expected for larger pressure gradients. On the other hand, the variable term in the expression for the recovery factor (eq. (67)) is proportional to the square of the Mach number and would be unimportant at low speeds. It is therefore not surprising that recovery factors obtained by the present method do not agree with those obtained in reference 17, where the variation of fluid properties was neglected.

Velocity profiles at the midchord point $(x = 1)$ of the two wings are presented in figure 11. The effect of heat transfer on the local velocity in the boundary layer is seen to be quite large - there is a marked thinning of the boundary layer when heat is extracted and a thickening when heat is added. Although the local velocity and its first derivative are altered only slightly because of the pressure gradient, the local curvature of the profiles appears to be affected to a greater extent. In particular, the profiles for zero heat transfer and a hot wall have an inflection point when the pressure gradient is adverse; whereas no inflection point is evident when the pressure gradient is favorable, even when the wall temperature is four times the ambient air temperature. In general, a velocity profile without an inflection point indicates greater laminar stability than one having an inflection point. (The shape of the temperature profile, however, also affects the criterion of stability.)

Although the local velocity near the outer edge of the boundary layer did not exceed the free-stream velocity, as discussed in reference 18, the functions $g'(\eta)$ are of a form indicating that such an overshoot may exist for slightly larger pressure gradients. (See fig. 1.)

Temperature profiles for the example wings are plotted in figure 12. These profiles do not differ greatly with the two different pressure gradients. The effect of heat transfer is quite large, however, as is evident from a comparison of the extremely thin profiles when the wall temperature ratio is 0.25 with the relatively thick profiles when this ratio is 4.

The ratio of the displacement thickness along the example wings δ^* to the displacement thickness along an equivalent flat plate δ^*_{FP} is plotted in figure 13. The displacement thickness is found to be less than the flat plate value for the adverse pressure gradient and greater for the favorable pressure gradient. This behavior is opposite to the trend found for incompressible flow and can be explained as follows: The ratio of displacement thicknesses is found to be

$$\frac{\delta^*}{\delta^*_{FP}} = 1 + \frac{\gamma}{2} M_r^2 \, \varepsilon \, a_N x^N + \frac{\varepsilon \, a_N x^N \left[\alpha_N + (\gamma-1) \, M_r^2 \, B_N\right]}{1.72 + 4.02 \, K + (\gamma-1)(1.11) \, M_r^2} \tag{76}$$

For incompressible flow and zero heat transfer, equation (76) reduces to

$$\frac{\delta^*}{\delta^*_{FP}} = 1 - 2.6\ \varepsilon\ a_N x^N \tag{77}$$

In a favorable gradient ($\varepsilon\ a_N x^N$ positive), δ^*/δ^*_{FP} as expressed by equation (77) decreases; whereas the ratio increases in an adverse gradient. This well-known thinning or thickening of the boundary layer is essentially an effect of the change in local Reynolds number caused by the change in the external velocity. 2968

As the Mach number is increased, however, the term $\frac{\gamma}{2} M_r^2\ \varepsilon\ a_N x^N$ in equation (76) becomes of importance. This term is related to the change in density at the outer edge of the boundary layer. Its significance may be qualitatively determined by supposing for the moment that viscosity may be neglected and by consideration of the two-dimensional compressible vorticity transport equation for an inviscid fluid

$$\frac{D}{Dt}\left(\frac{\Omega}{\rho}\right) = 0 \tag{78}$$

This expression shows that the vorticity changes in the same sense as the density. In a favorable pressure gradient, therefore, the vorticity will decrease along the wing because the density decreases along the wing. A decrease of vorticity in the boundary layer tends to thicken this layer.

If the complete equation for a viscous fluid is considered, there may be two opposing effects which occur at high Mach numbers: The effect of a favorable pressure gradient on Reynolds number (and hence viscosity) tends to thin the boundary layer; at the same time, the effect of the favorable pressure gradient on the vorticity directly tends to thicken the boundary layer. (A similar argument applies to adverse pressure gradients.) At a sufficiently high Mach number this second effect will predominate, as was found in the case of the present examples. For the case of constant pressure gradients and zero heat transfer, it can be shown that the aforementioned reversal of trends in the function δ^*/δ^*_{FP} occurs at a Mach number of 1.76.

If the Mach number is further increased, the thickening or thinning of the boundary layer will also affect the slope of the velocity profiles at the wall and hence the skin friction. For small constant pressure gradients and zero heat transfer, the skin friction trends are found to reverse at a Mach number of 4.71.

The effect of Prandtl number on the local friction drag parameter over the wing with the adverse pressure gradient is shown in figure 14. At midchord, a Prandtl number of 1 yields a friction drag coefficient 4 percent lower than a Prandtl number of 0.72 when the wall is insulated. A solution for a Prandtl number of 1 for flows with heat transfer was not obtained, but it is expected that the effect would be considerably larger than the 4 percent found for flows with zero heat transfer.

As a check on the accuracy of the present method, the solution for Prandtl number 1 and zero heat transfer was compared with an exact solution of Howarth (refs. 13 and 14), which applies even at pressure gradients as large as required for separation. At the midchord station the local friction drag parameter agrees within 0.7 percent with that obtained by Howarth.

CONCLUDING REMARKS

A method for the calculation of compressible laminar boundary layer characteristics for flows with heat transfer and small arbitrary pressure gradients is presented. This method was applied to the flow over two representative wings - one with a constant adverse pressure gradient, the other with a constant favorable pressure gradient. The investigation led to the following conclusions: It was found that the deviations in skin friction caused by the pressure gradient were magnified when the wall was heated and reduced when the wall was cooled. Large amounts of cooling were found to reverse the rate of change of skin friction along the wing caused by a pressure gradient alone.

Local rates of heat transfer were found to vary in direct opposition to the skin friction: If the pressure gradient was such that the shearing stress decreased along the wing, then the heat-transfer rate increased, and vice versa.

Temperature recovery factors were found to be affected by the pressure gradient. The percentage change in recovery factor along the wing was somewhat smaller than the percentage change in the external velocity.

The displacement thickness at a Mach number of 3 was found to be greater than the displacement thickness of an equivalent flat plate when the pressure gradient is favorable and less than the flat plate displacement thickness for the adverse pressure gradient. This result is opposite to the trend found at low speeds.

Lewis Flight Propulsion Laboratory
National Advisory Committee for Aeronautics
Cleveland, Ohio, August 19, 1953

APPENDIX A

SYMBOLS

The following symbols are used in this report:

$A_1, A_2, \ldots$	arbitrary constants (eq. (C2))
a_N	a measure of the shape of the external velocity distribution
B_N	function appearing in equation (71)
$b_1, b_2, \ldots$	arbitrary constants (eq. (76))
C	constant of proportionality in viscosity-temperature relation
C_F	average friction drag coefficient $= \frac{1}{\frac{1}{2} \rho_r u_r^2 x} \int_0^x \tau_w \, dx$
c_f	local friction drag coefficient $= \frac{\tau_w}{\frac{1}{2} \rho_r u_r^2}$
c_p	specific heat at constant pressure
F_R	temperature recovery factor
f	solution of zero-order momentum equation
G	function defined in equations (C4) and (C7)
g	solution of first-order momentum equation
H	solution of first-order energy equation with heat transfer
h	solution of first-order energy equation without heat transfer
K	factor describing heat-transfer conditions (eq. (34))
k	thermal conductivity
M	Mach number
N	exponent in free-stream velocity distribution, $(u_e^* = 1 + \varepsilon a_N x^N)$
Nu	Nusselt number $= \frac{C \, t_w^* \, x}{t_{aw}^* - t_w^*} \left.\frac{\partial t^*}{\partial y}\right)_w$
n	transformed variable
Pr	Prandtl number $= \mu \, c_p / k$

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p	static pressure
q	local rate of heat transfer
R	gas constant
Re	Reynolds number = $u_r\, x/\nu_r$
r	solution of zero-order energy equation
S	Sutherland's constant
s	solution of zero-order energy equation
T	total temperature
t	static temperature
u	velocity in x-direction
v	velocity in y-direction
x	distance along surface measured from leading edge
Y	normal coordinate of surface
y	distance from surface measured perpendicular to surface
$\left.\begin{matrix}\alpha_N\\ \beta_N\end{matrix}\right\}$	functions appearing in equation (71)
γ	ratio of specific heats
δ^*	displacement thickness
ε	small quantity - a measure of magnitude of velocity distribution at edge of boundary layer
η	characteristic variable defined by equation (26)
θ	dummy variable
μ	coefficient of viscosity
ν	kinematic viscosity = μ/ρ
ξ	dummy variable

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CR-4 back

ρ mass density

τ shearing stress

φ transformed stream function

ψ stream function

Ω vorticity, $\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$ 2968

Subscripts:

aw adiabatic wall

e conditions at outer edge of boundary layer

FP equivalent flat-plate value

r reference condition

w conditions at wall or surface

∞ undisturbed free-stream condition

x,y,n partial differentiation with respect to x, y, or n

M value of function corresponding to given value of M

N value of function corresponding to given value of N

Superscripts:

$*$ dimensionless quantities defined by equation (11)

$'$ differentiation with respect to η

Special Notation

A bar over a quantity indicates the order of approximation. (A single bar signifies a zero-order quantity, double bar signifies a first-order quantity, etc.)

A repeated index N appearing on a and one or more other symbols indicates summation: $\left[1 + \varepsilon\ a_N x^N \equiv 1 + \varepsilon(a_1 x + a_2 x^2 + \ldots)\right]$.

The symbol I preceding a quantity indicates integration from zero to η: $\left[\text{for example, } Ir(\eta) = \int_0^\eta r(\xi)d\xi\right]$.

APPENDIX B

NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

By Lynn U. Albers

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Each of the ordinary differential equations for f, r, s, g, h, and H with its associated boundary conditions at zero and infinity constitutes a two-point boundary value problem. With the exception of the Blasius equation all equations are linear, and the principle of superpositions of any two solutions may be used to satisfy the boundary conditions at infinity. Usually, two solutions close to the correct one were used in the final combination in order to minimize round-off errors. All integrations were performed on the IBM Card-Programmed Electronic Calculator. The combination of solutions and rounding to four decimal places was accomplished on the IBM Type 604 Calculating Punch by using general purpose floating-point control panels.

The integration technique will be described for the g problem, but it will be applicable to all the other problems with slight modifications. If $g'''(\eta)$ is given at five values of η, a fourth-degree polynomial in η may be passed through the set of values; and if g, g', and g'' are known at the fifth point, the polynomial representation of g''' may be integrated to yield g, g', and g'' at the next (sixth) point. These quantities may then be substituted in the differential equation (41) to yield g''' at the sixth point. Thus, by using the five previous points, the integration may be extended one step at a time.

The integration was initiated with an assumed trial value of $g''(0)$ and a value of $g'''(0)$ calculated from the equation. This value of $g'''(0)$ was also used as a first estimate of g''' at the next four points. The fourth-degree polynomial representing g''' over this range was then integrated to yield g, g', and g'' at the second point. Substitution in the equation then yielded a better estimate of g''' at the second point. Integration of the fourth-degree polynomial representation of g''' from zero to successive points was alternated with substitution in the equation to improve values of g''' in an iterative fashion until convergence was obtained at the five initial points.

It was found that when g' was close to its boundary value at infinity, the regular integration process encountered oscillations in the function g'''. To avoid this phenomenon, a procedure analogous to the starting procedure was used in an iterative manner. This smoothing process was used from $\eta = 3.4$ on. Integration was carried to a point which would yield four-decimal-point accuracy in the value of $g''(0)$ and in the g' and g'' data.

All integrations were performed using a step size of 0.1. Subsequent investigation of the effect of step size indicated that tabular values of the functions f, r, and s are correct as presented in table II, while the functions g, h, and H may be in error by 1 in the fourth place.

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APPENDIX C

SOLUTION OF FIRST ORDER-EQUATIONS FOR $N = 0$

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Physically, the solution of the first-order equations for $N = 0$ is of little interest because the external flow represented by $u_e^* = 1 + \varepsilon$ is simply the flow over a flat plate; the term ε arises because the reference velocity is taken slightly different from the stream velocity. In practical applications, flat-plate flows would be handled by the zero-order solutions. The case of $N = 0$ may be of academic interest, however, in addition to supplying limiting conditions for cases of $N \neq 0$.

The first-order momentum equation (eq. (38)) for $N = 0$ becomes

$$g_{01}''' + f\, g_{01}'' + f'' g_{01} = 0 \tag{C1}$$

$$g_{01}(0) = g_{01}'(0) = 0$$

$$g_{01}'(\infty) = 1$$

The functions g_{02} and g_{03} vanish identically. The general solution of this equation is

$$g_0(\eta) = A_1 f' + A_2(f + f'\eta) + A_3 \left[(f + f'\eta) \int_0^\eta \frac{f'f''\, d\eta}{(2f'^2 - ff'')^2} - f' \int_0^\eta \frac{(f + f'\eta)f''\, d\eta}{(2f'^2 - ff'')^2}\right] \tag{C2}$$

The coefficient of A_2 in equation (C2) was given in reference 19. From the boundary conditions it can be found that $A_1 = A_3 = 0$ and $A_2 = \frac{1}{4}$. Therefore,

$$g_{01}(\eta) = \frac{1}{4}\,(f + f'\eta) \tag{C3}$$

The first-order energy equation for $N = 0$ and zero heat transfer reduces to

$$h_{01}'' + Pr\, f\, h_{01}' = Pr\,(g_{01} r' + f''\, g_{01}'') = G_1(\eta) \tag{C4}$$

$$h'_{01}(0) = 0 \qquad h_{01}(\infty) = 1$$

Equation (C4) is a first-order linear equation in h'_0. The solution satisfying the boundary conditions is

$$h_{01}(\eta) = 1 - \int_{\eta}^{\infty} [f''(\xi)]^{Pr} \int_{0}^{\xi} \frac{G_1(\theta)}{[f''(\theta)]^{Pr}} \, d\theta \, d\xi \tag{C5}$$

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The function h_{02} is identically equal to zero. For flows with arbitrary rates of heat transfer, the following equations arise for $N = 0$:

$$H''_{01} + Pr\ f\ H'_{01} = Pr \left[g_0 r' + f''\ g''_0\right] = G_1(\eta) \tag{C6}$$

$$H''_{04} + Pr\ f\ H'_{04} = \frac{2\ Pr}{\gamma-1}\ g_0 s' = G_4(\eta) \tag{C7}$$

$$H_0(0) = 0 \qquad H_{01}(\infty) = 1 \qquad H_{04}(\infty) = 0$$

$$H_{02} = H_{03} = H_{05} \equiv 0$$

The solution of equation (C6) satisfying the appropriate boundary conditions is

$$H_{01}(\eta) = 1 - \int_{\eta}^{\infty} [f''(\xi)]^{Pr} \int_{0}^{\xi} [f''(\theta)]^{-Pr}\ G_1(\theta)\ d\theta\ d\xi\ -$$

$$\frac{1 - \int_{0}^{\infty} [f''(\xi)]^{Pr} \int_{0}^{\xi} [f''(\theta)]^{-Pr}\ G_1(\theta)\ d\theta\ d\xi}{\int_{0}^{\infty} [f''(\xi)]^{Pr}\ d\xi} \int_{\eta}^{\infty} [f''(\xi)]^{Pr}\ d\xi \tag{C8}$$

Similarly, the solution of equation (C7) is

$$H_{04}(\eta) = -\int_{\eta}^{\infty}[f''(\xi)]^{Pr}\int_{0}^{\xi}[f''(\theta)]^{-Pr}G_4(\theta)\,d\theta\,d\xi +$$

$$\frac{\int_{0}^{\infty}[f''(\xi)]^{Pr}\int_{0}^{\xi}[f''(\theta)]^{-Pr}G_4(\theta)\,d\theta\,d\xi}{\int_{0}^{\infty}[f''(\xi)]^{Pr}\,d\xi}\int_{\eta}^{\infty}[f''(\xi)]^{Pr}\,d\xi \tag{C9}$$

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The function $H_0(\eta)$ is again obtained by a linear combination of $H_{01}(\eta)$ and $H_{04}(\eta)$ in the following manner:

$$H_0(\eta) = H_{01}(\eta) + \frac{K}{M_r^2}H_{04}(\eta) \tag{C10}$$

Values of $h_0(0)$ and $H_0'(0)$ for $Pr = 0.72$ were obtained by numerical integration and are listed in table I.

APPENDIX D

SOLUTION FOR $Pr = 1$

In order to establish the effect of Prandtl number on skin friction and to provide a basis for comparison with other solutions, some of the energy equations were solved for the special case of $Pr = 1$. The solution of the zero-order energy equation for $Pr = 1$ is

$$r(\eta) = 1 - \frac{1}{4}\,(f')^2 \tag{D1}$$

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$$s(\eta) = 2 - f'(\eta) \tag{D2}$$

The solution of the first-order energy equations for zero heat transfer is, for $Pr = 1$,

$$\left.\begin{aligned} h_{N1}(\eta) &= \frac{1}{2}\,f'\ g'_{N1} \\ &\text{and} \\ h_{N2}(\eta) &= \frac{1}{2}\,f'\ g'_{N2} \end{aligned}\right\} \tag{D3}$$

The linear combination of equations (D3) yields

$$h_N(\eta) = \frac{1}{2}\,f'\left[g'_{N1} + M_r^2\ g'_{N2}\right] \tag{D4}$$

The function g_{N1} is independent of Prandtl number, and hence the values appearing in table III apply for all Prandtl numbers. The function g_{12} was calculated numerically for a Prandtl number of 1, and results of this calculation appear in table VII.

The complete solution of the first-order energy equation for $Pr = 1$ and flows with arbitrary rates of heat transfer was not found. The following are the solutions of equations (45) and (46) for $Pr = 1$ and heat transfer:

$$\left.\begin{aligned} H_{N1}(\eta) &= \frac{1}{2}\,f'\ g'_{N1} \\ H_{N2}(\eta) &= \frac{1}{2}\,f'\ g'_{N2} \end{aligned}\right\} \tag{D5}$$

and for $Pr = 1$ and $N = 0$:

$$H_{04}(\eta) = \frac{\eta\ f''(\eta)}{2(\gamma - 1)} \tag{D6}$$

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TABLE I. - INITIAL VALUES

[Pr, 0.72; γ, 1.4]

$f''(0) = 1.3282$ $r(0) = 0.8477$

$s(0) = 2.0748$ $s'(0) = -1.2267$

	N = 0	N = 1	N = 2	N = 3
$g''_{N1}(0)$	0.9962	4.0821	6.3546	8.2879
$g''_{N2}(0)$	0	.2807	.5847	.8717
$g''_{N3}(0)$	0	5.0447	8.9738	12.4065
$h_{N1}(0)$	0.1523	0.1524	0.1526	0.1528
$h_{N2}(0)$	0	.0085	.0123	.0146
$H'_{N1}(0)$	0.0904	0.1479	0.1802	0.2042
$H'_{N2}(0)$	0	.0082	.0145	.0195
$H'_{N3}(0)$	0	-.4201	-.2200	.0899
$H'_{N4}(0)$	1.5326	5.5574	8.6985	11.3879
$H'_{N5}(0)$	0	5.3452	10.1820	14.5535

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TABLE II. - SOLUTIONS OF ZERO-ORDER MOMENTUM AND ENERGY EQUATIONS

[Pr, 0.72]

η	f	f'	f''	Ir	r	r'	Is	s	s'
0	0.0000	0.0000	1.3282	0.0000	0.8477	0.0000	0.0000	2.0740	-1.2267
.1	.0066	.1328	1.3279	.0047	.8445	-.0635	.2013	1.9581	-1.2266
.2	.0266	.2655	1.3259	.1667	.8350	-.1268	.3904	1.8295	-1.2252
.3	.0597	.3979	1.3203	.2514	.8192	-.1893	.5672	1.7071	-1.2215
.4	.1061	.5294	1.3096	.3323	.7972	-.2503	.7319	1.5853	-1.2143
.5	.1656	.6596	1.2920	.4107	.7692	-.3066	.8843	1.4644	-1.2026
.6	.2379	.7875	1.2663	.4860	.7366	-.3686	1.0246	1.3450	-1.1853
.7	.3230	.9128	1.2314	.5676	.6969	-.4110	1.1534	1.2276	-1.1617
.8	.4203	1.0335	1.1860	.6252	.6536	-.4522	1.2704	1.1189	-1.1311
.9	.5295	1.1495	1.1317	.6882	.6067	-.4846	1.3761	1.0016	-1.0932
1.0	.6500	1.2595	1.0670	.7464	.5570	-.5071	1.4709	.8945	-1.0476
1.1	.7812	1.3626	.9954	.7996	.5057	-.5189	1.5552	.7983	-.9958
1.2	.9223	1.4579	.9124	.8476	.4536	-.5198	1.6295	.6957	-.9361
1.3	1.0785	1.5449	.8269	.8903	.4020	-.5101	1.6945	.6052	-.8713
1.4	1.2310	1.6230	.7361	.9260	.3519	-.4906	1.7508	.5215	-.8020
1.5	1.3968	1.6921	.6455	.9608	.3042	-.4627	1.7990	.4449	-.7296
1.6	1.5691	1.7522	.5566	.9890	.2596	-.4288	1.8400	.3757	-.6550
1.7	1.7469	1.8035	.4715	1.0128	.2187	-.3889	1.8744	.3138	-.5820
1.8	1.9295	1.8467	.3934	1.0326	.1819	-.3470	1.9030	.2592	-.5099
1.9	2.1160	1.8882	.3205	1.0494	.1496	-.3048	1.9266	.2117	-.4406
2.0	2.3057	1.9110	.2569	1.0689	.1211	-.2628	1.9456	.1709	-.3759
2.1	2.4980	1.9339	.2081	1.0737	.0969	-.2228	1.9609	.1364	-.3168
2.2	2.6924	1.9517	.1569	1.0824	.0765	-.1854	1.9730	.1075	-.2623
2.3	2.8882	1.9654	.1179	1.0891	.0597	-.1522	1.9826	.0837	-.2146
2.4	3.0853	1.9756	.0875	1.0944	.0459	-.1231	1.9899	.0644	-.1730
2.5	3.2833	1.9831	.0636	1.0964	.0349	-.0961	1.9956	.0489	-.1376
2.6	3.4819	1.9885	.0454	1.1014	.0262	-.0770	1.9998	.0366	-.1078
2.7	3.6809	1.9923	.0317	1.1037	.0194	-.0595	2.0030	.0271	-.0833
2.8	3.8803	1.9950	.0217	1.1054	.0142	-.0464	2.0053	.0198	-.0635
2.9	4.0799	1.9967	.0146	1.1066	.0102	-.0341	2.0070	.0143	-.0477
3.0	4.2796	1.9979	.0096	1.1075	.0073	-.0252	2.0083	.0102	-.0353
3.1	4.4794	1.9987	.0058	1.1081	.0051	-.0184	2.0091	.0072	-.0257
3.2	4.6793	1.9992	.0039	1.1085	.0035	-.0132	2.0097	.0050	-.0185
3.3	4.8793	1.9995	.0024	1.1086	.0024	-.0094	2.0101	.0034	-.0131
3.4	5.0793	1.9997	.0015	1.1090	.0016	-.0065	2.0104	.0023	-.0092
3.5	5.2793	1.9998	.0009	1.1091	.0011	-.0045	2.0106	.0015	-.0063
3.6	5.4793	1.9999	.0005	1.1092	.0007	-.0031	2.0107	.0010	-.0043
3.7	5.6793	2.0000	.0003	1.1093	.0004	-.0020	2.0108	.0006	-.0026
3.8	5.8793	2.0000	.0002	1.1093	.0003	-.0013	2.0108	.0004	-.0018
3.9	6.0793	2.0000	.0001	1.1093	.0002	-.0009	2.0108	.0003	-.0012
4.0	6.2792	2.0000	.0000	1.1093	.0001	-.0005	2.0109	.0002	-.0008
4.1	6.4792	2.0000	.0000	1.1093	.0001	-.0003	2.0109	.0001	-.0005
4.2	6.6792	2.0000	.0000	1.1093	.0000	-.0002	2.0109	.0001	-.0003
4.3	6.8792	2.0000	.0000	1.1093	.0000	-.0001	2.0109	.0000	-.0002
4.4	7.0792	2.0000	.0000	1.1094	.0000	-.0001	2.0109	.0000	-.0001
4.5	7.2792	2.0000	.0000	1.1094	.0000	-.0001	2.0109	.0000	-.0001

TABLE III. - SOLUTIONS OF FIRST-ORDER MOMENTUM EQUATION

[Pr, 0.72; γ, 1.40]

η	g_{11}	g_{12}	g_{13}	g'_{11}	g'_{12}	g'_{13}	g''_{11}	g''_{12}	g''_{13}	N
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	4.0821	0.2807	5.0447	1
.1	.0197	.0013	.0839	.3882	.0247	.4638	3.6821	.2132	4.2394	1
.2	.0763	.0047	.0902	.7364	.0427	.8495	3.2821	.1474	3.4831	1
.3	.1657	.0096	.1913	1.0446	.0543	1.1620	2.8823	.0851	2.7758	1
.4	.2839	.0154	.3203	1.3189	.0599	1.4063	2.4829	.0279	2.1176	1
.5	.4269	.0214	.4705	1.5413	.0601	1.5872	2.0849	-.0225	1.5089	1
.6	.5908	.0272	.6358	1.7300	.0557	1.7098	1.6903	-.0647	.9505	1
.7	.7716	.0324	.8106	1.8795	.0475	1.7790	1.3018	-.0977	.4438	1
.8	.9655	.0366	.9900	1.9907	.0365	1.8003	.9238	-.1207	-.0089	1
.9	1.1685	.0397	1.1693	2.0648	.0237	1.7791	.5619	-.1334	-.4058	1
1.0	1.3773	.0413	1.3446	2.1038	.0101	1.7213	.2228	-.1360	-.7422	1
1.1	1.5882	.0417	1.5125	2.1104	-.0032	1.6328	-.0862	-.1295	-1.0171	1
1.2	1.7984	.0407	1.6703	2.0878	-.0155	1.5200	-.3577	-.1152	-1.2282	1
1.3	2.0050	.0386	1.8159	2.0403	-.0261	1.3893	-.5851	-.0951	-1.3752	1
1.4	2.2057	.0356	1.9478	1.9724	-.0344	1.2470	-.7635	-.0712	-1.4597	1
1.5	2.3989	.0318	2.0651	1.8893	-.0403	1.0993	-.8904	-.0458	-1.4856	1
1.6	2.5833	.0276	2.1677	1.7961	-.0436	.9516	-.9658	-.0211	-1.4591	1
1.7	2.7580	.0232	2.2556	1.6978	-.0446	.8089	-.9928	.0018	-1.3884	1
1.8	2.9228	.0188	2.3297	1.5990	-.0435	.6751	-.9770	.0198	-1.2832	1
1.9	3.0779	.0146	2.3911	1.5036	-.0408	.5531	-.9258	.0338	-1.1540	1
2.0	3.2237	.0107	2.4408	1.4147	-.0369	.4448	-.8482	.0431	-1.0110	1
2.1	3.3611	.0072	2.4805	1.3345	-.0323	.3510	-.7533	.0479	-.8637	1
2.2	3.4910	.0042	2.5115	1.2643	-.0274	.2719	-.6497	.0488	-.7200	1
2.3	3.6143	.0017	2.5353	1.2046	-.0226	.2067	-.5450	.0466	-.5861	1
2.4	3.7322	-.0003	2.5533	1.1551	-.0182	.1542	-.4451	.0424	-.4661	1
2.5	3.8457	-.0019	2.5666	1.1153	-.0142	.1129	-.3543	.0368	-.3624	1
2.6	3.9556	-.0032	2.5762	1.0839	-.0108	.0811	-.2750	.0308	-.2755	1
2.7	4.0627	-.0041	2.5830	1.0598	-.0080	.0572	-.2083	.0249	-.2050	1
2.8	4.1677	-.0048	2.5878	1.0418	-.0058	.0397	-.1541	.0195	-.1493	1
2.9	4.2712	-.0053	2.5911	1.0286	-.0041	.0270	-.1113	.0148	-.1064	1
3.0	4.3736	-.0056	2.5933	1.0192	-.0028	.0180	-.0786	.0109	-.0744	1
3.1	4.4751	-.0059	2.5948	1.0126	-.0019	.0118	-.0542	.0078	-.0509	1
3.2	4.5762	-.0060	2.5958	1.0081	-.0013	.0076	-.0366	.0054	-.0342	1
3.3	4.6768	-.0061	2.5964	1.0051	-.0008	.0048	-.0241	.0037	-.0224	1
3.4	4.7772	-.0062	2.5968	1.0032	-.0005	.0030	-.0155	.0024	-.0144	1
3.5	4.8775	-.0062	2.5970	1.0019	-.0003	.0018	-.0098	.0016	-.0092	1
3.6	4.9776	-.0063	2.5971	1.0011	-.0002	.0011	-.0061	.0010	-.0057	1
3.7	5.0777	-.0063	2.5972	1.0007	-.0001	.0007	-.0037	.0006	-.0035	1
3.8	5.1778	-.0063	2.5973	1.0004	-.0001	.0004	-.0022	.0003	-.0021	1
3.9	5.2778	-.0063	2.5973	1.0002	.0000	.0002	-.0013	.0002	-.0012	1
4.0	5.3778	-.0063	2.5973	1.0001	.0000	.0001	-.0007	.0001	-.0007	1
4.1	5.4778	-.0063	2.5973	1.0001	.0000	.0001	-.0004	.0001	-.0004	1
4.2	5.5778	-.0063	2.5973	1.0000	.0000	.0000	-.0002	.0000	-.0002	1
4.3	5.6778	-.0063	2.5973	1.0000	.0000	.0000	-.0001	.0000	-.0001	1
4.4	5.7778	-.0063	2.5973	1.0000	.0000	.0000	-.0001	.0000	-.0001	1
4.5	5.8778	-.0063	2.5973	1.0000	.0000	.0000	.0000	.0000	.0000	1

TABLE III. - Continued. SOLUTIONS OF FIRST-ORDER MOMENTUM EQUATION

[Pr, 0.72; γ, 1.40]

η	g_{21}	g_{22}	g_{23}	g'_{21}	g'_{22}	g'_{23}	g''_{21}	g''_{22}	g''_{23}	N
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	6.3546	0.5847	8.9738	2
.1	.0304	.0027	.0421	.5955	.0517	.8161	5.5873	.4499	7.3667	2
.2	.1165	.0099	.1581	1.1119	.0901	1.4772	4.7743	.3197	5.8766	2
.3	.2503	.0203	.3328	1.5512	.1159	1.9957	4.0167	.1981	4.5152	2
.4	.4248	.0327	.5589	1.9164	.1301	2.3848	3.2917	.0884	3.2878	2
.5	.6312	.0460	.8059	2.2108	.1341	2.6577	2.6041	-.0069	2.1942	2
.6	.8642	.0592	1.0810	2.4385	.1293	2.8880	1.9573	-.0858	1.2316	2
.7	1.1168	.0716	1.3685	2.6037	.1175	2.9083	1.3538	-.1471	.3947	2
.8	1.3830	.0826	1.6601	2.7108	.1005	2.9109	.7964	-.1903	-.3219	2
.9	1.6572	.0916	1.9485	2.7647	.0801	2.8477	.2886	-.2157	-.9233	2
1.0	1.9343	.0985	2.2278	2.7703	.0579	2.7299	-.1654	-.2243	-1.4140	2
1.1	2.2098	.1032	2.4930	2.7335	.0357	2.5685	-.5604	-.2181	-1.7979	2
1.2	2.4798	.1057	2.7404	2.6604	.0147	2.3738	-.8914	-.1994	-2.0786	2
1.3	2.7409	.1062	2.9670	2.5575	-.0039	2.1560	-1.1542	-.1712	-2.2607	2
1.4	2.9906	.1050	3.1711	2.4319	-.0193	1.9247	-1.3461	-.1370	-2.3501	2
1.5	3.2268	.1025	3.3518	2.2907	-.0312	1.6888	-1.4670	-.0999	-2.3549	2
1.6	3.4484	.0989	3.5090	2.1408	-.0393	1.4562	-1.5200	-.0633	-2.2857	2
1.7	3.6549	.0947	3.6434	1.9887	-.0439	1.2337	-1.5113	-.0296	-2.1553	2
1.8	3.8463	.0902	3.7563	1.8403	-.0454	1.0267	-1.4498	-.0008	-1.9780	2
1.9	4.0232	.0857	3.8494	1.7002	-.0443	.8392	-1.3465	.0218	-1.7690	2
2.0	4.1867	.0814	3.9248	1.5780	-.0413	.6735	-1.2136	.0379	-1.5431	2
2.1	4.3381	.0775	3.9849	1.4581	-.0369	.5307	-1.0632	.0477	-1.3137	2
2.2	4.4788	.0741	4.0317	1.3596	-.0319	.4105	-.9065	.0520	-1.0922	2
2.3	4.6105	.0711	4.0677	1.2767	-.0267	.3117	-.7589	.0518	-.8872	2
2.4	4.7347	.0687	4.0947	1.2086	-.0216	.2323	-.6097	.0484	-.7044	2
2.5	4.8587	.0668	4.1147	1.1542	-.0171	.1700	-.4816	.0429	-.5468	2
2.6	4.9659	.0653	4.1292	1.1117	-.0131	.1221	-.3713	.0365	-.4153	2
2.7	5.0754	.0642	4.1395	1.0793	-.0098	.0861	-.2796	.0298	-.3087	2
2.8	5.1820	.0633	4.1467	1.0552	-.0071	.0596	-.2057	.0235	-.2246	2
2.9	5.2866	.0627	4.1517	1.0376	-.0051	.0405	-.1478	.0180	-.1600	2
3.0	5.3897	.0623	4.1550	1.0252	-.0035	.0270	-.1039	.0133	-.1118	2
3.1	5.4918	.0620	4.1572	1.0165	-.0024	.0177	-.0714	.0096	-.0764	2
3.2	5.5931	.0618	4.1587	1.0106	-.0016	.0114	-.0481	.0067	-.0514	2
3.3	5.6940	.0617	4.1596	1.0066	-.0010	.0072	-.0314	.0045	-.0336	2
3.4	5.7945	.0616	4.1608	1.0041	-.0006	.0045	-.0202	.0030	-.0217	2
3.5	5.8948	.0616	4.1605	1.0025	-.0004	.0027	-.0127	.0019	-.0138	2
3.6	5.9950	.0615	4.1607	1.0015	-.0002	.0016	-.0079	.0012	-.0086	2
3.7	6.0951	.0615	4.1609	1.0008	-.0001	.0010	-.0047	.0007	-.0052	2
3.8	6.1952	.0615	4.1609	1.0005	-.0001	.0006	-.0028	.0004	-.0032	2
3.9	6.2952	.0615	4.1610	1.0003	.0000	.0003	-.0016	.0002	-.0018	2
4.0	6.3953	.0615	4.1610	1.0001	.0000	.0002	-.0009	.0001	-.0011	2
4.1	6.4953	.0615	4.1610	1.0001	.0000	.0001	-.0005	.0001	-.0006	2
4.2	6.5953	.0615	4.1610	1.0000	.0000	.0000	-.0003	.0000	-.0004	2
4.3	6.6953	.0615	4.1610	1.0000	.0000	.0000	-.0001	.0000	-.0002	2
4.4	6.7953	.0615	4.1610	1.0000	.0000	.0000	-.0001	.0000	-.0001	2
4.5	6.8953	.0615	4.1610	1.0000	.0000	.0000	.0000	.0000	.0000	2

TABLE III. - Concluded. SOLUTIONS OF FIRST-ORDER MOMENTUM EQUATION

[Pr, 0.72; γ, 1.40]

η	g_{31}	g_{32}	g_{33}	g'_{31}	g'_{32}	g'_{33}	g''_{31}	g''_{32}	g''_{33}	N
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	8.2879	0.8717	12.4065	3
.1	.0394	.0040	.0579	.7690	.0770	1.1189	7.0947	.6698	10.0002	3
.2	.1499	.0147	.2161	1.4202	.1343	2.0067	5.9382	.4766	7.7924	3
.3	.3197	.0303	.4523	1.9587	.1729	2.6850	4.8442	.2986	5.8129	3
.4	.5381	.0488	.7469	2.3916	.1947	3.1772	3.8288	.1409	4.0726	3
.5	.7948	.0687	1.0823	2.7272	.2018	3.5073	2.8970	.0064	2.5677	3
.6	1.0806	.0887	1.4437	2.9739	.1968	3.6982	2.0527	-.1029	1.2856	3
.7	1.3869	.1077	1.8181	3.1405	.1881	3.7713	1.2933	-.1865	.2079	3
.8	1.7063	.1249	2.1947	3.2353	.1603	3.7460	.6163	-.2449	-.6847	3
.9	2.0319	.1396	2.5646	3.2665	.1339	3.6399	.0192	-.2794	-1.4106	3
1.0	2.3577	.1516	2.9805	3.2418	.1052	3.4689	-.4988	-.2922	-1.9855	3
1.1	2.6786	.1607	3.2567	3.1693	.0761	3.2474	-.9373	-.2861	-2.4220	3
1.2	2.9902	.1669	3.5687	3.0571	.0485	2.9887	-1.2945	-.2648	-2.7306	3
1.3	3.2890	.1704	3.8536	2.9132	.0236	2.7052	-1.5689	-.2313	-2.9202	3
1.4	3.5721	.1717	4.1093	2.7460	.0025	2.4083	-1.7600	-.1906	-3.0003	3
1.5	3.8377	.1711	4.3351	2.5639	-.0144	2.1084	-1.8696	-.1463	-2.9815	3
1.6	4.0847	.1690	4.5312	2.3747	-.0268	1.8149	-1.9022	-.1021	-2.8767	3
1.7	4.3186	.1659	4.6986	2.1858	-.0349	1.5365	-1.8658	-.0610	-2.7009	3
1.8	4.5220	.1621	4.8390	2.0035	-.0392	1.2765	-1.7712	-.0254	-2.4711	3
1.9	4.7137	.1581	4.9547	1.8330	-.0402	1.0425	-1.6315	.0033	-2.2051	3
2.0	4.8892	.1548	5.0484	1.6782	-.0388	.8361	-1.4606	.0245	-1.9204	3
2.1	5.0500	.1504	5.1229	1.5415	-.0356	.6585	-1.2725	.0385	-1.6331	3
2.2	5.1981	.1471	5.1810	1.4239	-.0313	.5091	-1.0798	.0460	-1.3566	3
2.3	5.3354	.1442	5.2256	1.3253	-.0265	.3864	-.8933	.0483	-1.1013	3
2.4	5.4638	.1418	5.2591	1.2448	-.0218	.2879	-.7208	.0466	-.8739	3
2.5	5.5849	.1398	5.2839	1.1805	-.0173	.2106	-.5676	.0422	-.6782	3
2.6	5.7004	.1383	5.3019	1.1305	-.0134	.1512	-.4364	.0365	-.5149	3
2.7	5.8114	.1371	5.3146	1.0925	-.0100	.1066	-.3277	.0301	-.3825	3
2.8	5.9192	.1363	5.3236	1.0642	-.0073	.0738	-.2406	.0241	-.2783	3
2.9	6.0245	.1356	5.3297	1.0437	-.0052	.0501	-.1725	.0184	-.1982	3
3.0	6.1281	.1352	5.3338	1.0292	-.0036	.0334	-.1211	.0139	-.1385	3
3.1	6.2305	.1349	5.3366	1.0191	-.0024	.0219	-.0830	.0098	-.0946	3
3.2	6.3320	.1347	5.3383	1.0122	-.0016	.0141	-.0560	.0073	-.0638	3
3.3	6.4330	.1346	5.3395	1.0076	-.0010	.0089	-.0363	.0043	-.0413	3
3.4	6.5336	.1345	5.3402	1.0047	-.0006	.0055	-.0233	.0029	-.0268	3
3.5	6.6340	.1344	5.3406	1.0028	-.0004	.0034	-.0147	.0020	-.0170	3
3.6	6.7342	.1344	5.3409	1.0017	-.0002	.0020	-.0091	.0012	-.0106	3
3.7	6.8343	.1344	5.3410	1.0010	-.0001	.0012	-.0058	.0007	-.0064	3
3.8	6.9344	.1344	5.3411	1.0005	-.0001	.0007	-.0033	.0004	-.0040	3
3.9	7.0345	.1344	5.3412	1.0003	.0000	.0004	-.0019	.0003	-.0023	3
4.0	7.1345	.1344	5.3412	1.0001	.0000	.0002	-.0011	.0001	-.0014	3
4.1	7.2345	.1344	5.3412	1.0001	.0000	.0001	-.0005	.0001	-.0007	3
4.2	7.3345	.1344	5.3412	1.0000	.0000	.0000	-.0003	.0001	-.0005	3
4.3	7.4345	.1344	5.3412	1.0000	.0000	.0000	-.0001	.0000	-.0002	3
4.4	7.5345	.1344	5.3412	1.0000	.0000	.0000	-.0001	.0000	-.0002	3
4.5	7.6345	.1344	5.3412	1.0000	.0000	.0000	.0000	.0000	.0000	3

TABLE IV. - SOLUTIONS OF FIRST-ORDER ENERGY EQUATION FOR ZERO HEAT TRANSFER

[Pr, 0.72; γ, 1.40]

η	Ih_{11}	Ih_{12}	h_{11}	h_{12}	h'_{11}	h'_{12}	N
0	0.0000	0.0000	0.1524	0.0085	0.0000	0.0000	1
.1	.0159	.0009	.1710	.0097	.3631	.0221	1
.2	.0353	.0020	.2232	.0126	.6712	.0348	1
.3	.0614	.0034	.3034	.0163	.9231	.0386	1
.4	.0968	.0053	.4059	.0200	1.1174	.0344	1
.5	.1432	.0074	.5249	.0230	1.2526	.0236	1
.6	.2021	.0098	.6544	.0246	1.3282	.0078	1
.7	.2742	.0123	.7886	.0244	1.3448	-.0111	1
.8	.3598	.0146	.9215	.0223	1.3051	-.0309	1
.9	.4583	.0167	1.0479	.0183	1.2139	-.0494	1
1.0	.5690	.0182	1.1628	.0126	1.0784	-.0648	1
1.1	.6904	.0191	1.2624	.0055	.9083	-.0755	1
1.2	.8208	.0193	1.3437	-.0024	.7151	-.0806	1
1.3	.9584	.0187	1.4051	-.0104	.5118	-.0799	1
1.4	1.1012	.0172	1.4460	-.0182	.3091	-.0737	1
1.5	1.2470	.0151	1.4673	-.0250	.1200	-.0629	1
1.6	1.3940	.0123	1.4708	-.0306	-.0470	-.0489	1
1.7	1.5406	.0090	1.4589	-.0347	-.1856	-.0331	1
1.8	1.6854	.0054	1.4347	-.0372	-.2925	-.0171	1
1.9	1.8273	.0016	1.4015	-.0382	-.3671	-.0021	1
2.0	1.9655	-.0022	1.3623	-.0377	-.4114	.0109	1
2.1	2.0996	-.0059	1.3201	-.0361	-.4291	.0214	1
2.2	2.2295	-.0094	1.2772	-.0335	-.4251	.0290	1
2.3	2.3551	-.0126	1.2356	-.0304	-.4044	.0338	1
2.4	2.4767	-.0155	1.1967	-.0269	-.3720	.0361	1
2.5	2.5946	-.0180	1.1614	-.0232	-.3326	.0363	1
2.6	2.7091	-.0201	1.1303	-.0197	-.2898	.0348	1
2.7	2.8208	-.0219	1.1035	-.0163	-.2467	.0321	1
2.8	2.9299	-.0234	1.0809	-.0133	-.2056	.0286	1
2.9	3.0371	-.0246	1.0622	-.0106	-.1679	.0248	1
3.0	3.1425	-.0255	1.0472	-.0083	-.1345	.0210	1
3.1	3.2466	-.0263	1.0352	-.0064	-.1057	.0173	1
3.2	3.3496	-.0268	1.0259	-.0049	-.0816	.0139	1
3.3	3.4519	-.0272	1.0187	-.0036	-.0619	.0110	1
3.4	3.5534	-.0276	1.0134	-.0026	-.0464	.0085	1
3.5	3.6546	-.0278	1.0094	-.0019	-.0338	.0064	1
3.6	3.7554	-.0279	1.0065	-.0013	-.0244	.0048	1
3.7	3.8559	-.0281	1.0044	-.0009	-.0173	.0035	1
3.8	3.9563	-.0281	1.0030	-.0006	-.0120	.0025	1
3.9	4.0565	-.0282	1.0020	-.0004	-.0082	.0018	1
4.0	4.1567	-.0282	1.0013	-.0003	-.0056	.0012	1
4.1	4.2568	-.0282	1.0008	-.0002	-.0037	.0008	1
4.2	4.3568	-.0283	1.0005	-.0001	-.0024	.0006	1
4.3	4.4569	-.0283	1.0003	-.0001	-.0016	.0004	1
4.4	4.5569	-.0283	1.0002	.0000	-.0010	.0002	1
4.5	4.6569	-.0283	1.0001	.0000	-.0006	.0002	1
4.6	4.7569	-.0283	1.0001	.0000	-.0004	.0001	1
4.7	4.8570	-.0283	1.0001	.0000	-.0002	.0001	1
4.8	4.9570	-.0283	1.0000	.0000	-.0001	.0000	1
4.9	5.0570	-.0283	1.0000	.0000	.0000	.0000	1

TABLE IV. - Continued. SOLUTIONS OF FIRST-ORDER ENERGY EQUATION FOR ZERO HEAT TRANSFER

[Pr, 0.72; γ, 1.40]

η	Ih_{21}	Ih_{22}	h_{21}	h_{22}	h_{21}'	h_{22}'	N
0	0.0000	0.0000	0.1526	0.0123	0.0000	0.0000	2
.1	.0162	.0013	.1812	.0148	.5533	.0465	2
.2	.0379	.0031	.2597	.0210	.9986	.0744	2
.3	.0695	.0056	.3774	.0291	1.3381	.0852	2
.4	.1143	.0089	.5239	.0375	1.5753	.0808	2
.5	.1749	.0130	.6892	.0449	1.7149	.0639	2
.6	.2525	.0178	.8638	.0500	1.7628	.0379	2
.7	.3477	.0230	1.0390	.0522	1.7267	.0061	2
.8	.4600	.0282	1.2067	.0512	1.6159	-.0275	2
.9	.5885	.0331	1.3600	.0468	1.4419	-.0595	2
1.0	.7314	.0374	1.4934	.0394	1.2181	-.0867	2
1.1	.8864	.0409	1.6025	.0297	.9597	-.1069	2
1.2	1.0510	.0433	1.6847	.0183	.6830	-.1185	2
1.3	1.2224	.0445	1.7390	.0063	.4044	-.1212	2
1.4	1.3979	.0446	1.7660	-.0056	.1392	-.1153	2
1.5	1.5747	.0434	1.7677	-.0165	-.0993	-.1023	2
1.6	1.7507	.0413	1.7474	-.0259	-.3013	-.0840	2
1.7	1.9236	.0383	1.7089	-.0332	-.4607	-.0627	2
1.8	2.0920	.0347	1.6568	-.0384	-.5753	-.0405	2
1.9	2.2547	.0307	1.5953	-.0414	-.6464	-.0191	2
2.0	2.4109	.0265	1.5288	-.0423	-.6782	-.0002	2
2.1	2.5604	.0223	1.4608	-.0415	-.6767	.0154	2
2.2	2.7031	.0183	1.3943	-.0393	-.6490	.0273	2
2.3	2.8394	.0145	1.3317	-.0362	-.6022	.0354	2
2.4	2.9696	.0111	1.2743	-.0324	-.5432	.0400	2
2.5	3.0944	.0080	1.2232	-.0283	-.4778	.0416	2
2.6	3.2145	.0054	1.1788	-.0241	-.4107	.0408	2
2.7	3.3304	.0032	1.1410	-.0202	-.3456	.0383	2
2.8	3.4429	.0014	1.1095	-.0165	-.2851	.0347	2
2.9	3.5525	-.0001	1.0838	-.0133	-.2307	.0304	2
3.0	3.6598	-.0013	1.0631	-.0104	-.1833	.0259	2
3.1	3.7653	-.0022	1.0469	-.0081	-.1431	.0215	2
3.2	3.8693	-.0029	1.0343	-.0061	-.1097	.0174	2
3.3	3.9722	-.0035	1.0247	-.0046	-.0828	.0138	2
3.4	4.0743	-.0039	1.0176	-.0034	-.0614	.0107	2
3.5	4.1758	-.0042	1.0123	-.0024	-.0448	.0081	2
3.6	4.2768	-.0044	1.0085	-.0017	-.0321	.0061	2
3.7	4.3776	-.0045	1.0058	-.0012	-.0227	.0044	2
3.8	4.4780	-.0046	1.0039	-.0008	-.0157	.0032	2
3.9	4.5783	-.0047	1.0026	-.0005	-.0107	.0022	2
4.0	4.6786	-.0047	1.0017	-.0004	-.0072	.0016	2
4.1	4.7787	-.0048	1.0011	-.0002	-.0048	.0011	2
4.2	4.8788	-.0048	1.0007	-.0001	-.0031	.0007	2
4.3	4.9788	-.0048	1.0004	-.0001	-.0020	.0005	2
4.4	5.0789	-.0048	1.0003	.0000	-.0014	.0003	2
4.5	5.1789	-.0048	1.0002	.0000	-.0008	.0002	2
4.6	5.2789	-.0048	1.0001	.0000	-.0005	.0002	2
4.7	5.3789	-.0048	1.0001	.0000	-.0003	.0001	2
4.8	5.4789	-.0048	1.0001	.0000	-.0002	.0001	2
4.9	5.5789	-.0048	1.0000	.0000	-.0001	.0000	2

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TABLE IV. - Concluded. SOLUTIONS OF FIRST-ORDER ENERGY EQUATION FOR ZERO HEAT TRANSFER

[Pr, 0.72; γ, 1.40]

η	Ih_{31}	Ih_{32}	h_{31}	h_{32}	h'_{31}	h'_{32}	N
0	0.0000	0.0000	0.1528	0.0146	0.0000	0.0000	3
.1	.0165	.0016	.1897	.0183	.7112	.0693	3
.2	.0400	.0038	.2897	.0275	1.2623	.1111	3
.3	.0761	.0072	.4371	.0396	1.6612	.1279	3
.4	.1286	.0118	.6172	.0524	1.9186	.1231	3
.5	.2001	.0176	.8165	.0637	2.0473	.1009	3
.6	.2921	.0244	1.0229	.0721	2.0616	.0663	3
.7	.4046	.0319	1.2255	.0767	1.9764	.0242	3
.8	.5368	.0396	1.4154	.0769	1.8073	-.0202	3
.9	.6870	.0471	1.5848	.0727	1.5710	-.0624	3
1.0	.8528	.0540	1.7279	.0646	1.2849	-.0987	3
1.1	1.0315	.0599	1.8407	.0534	0.9674	-.1261	3
1.2	1.2199	.0646	1.9209	.0398	0.6371	-.1429	3
1.3	1.4146	.0678	1.9683	.0251	0.3121	-.1486	3
1.4	1.6125	.0697	1.9841	.0104	0.0090	-.1440	3
1.5	1.8105	.0700	1.9713	-.0034	-.2584	-.1304	3
1.6	2.0059	.0691	1.9339	-.0154	-.4801	-.1104	3
1.7	2.1966	.0670	1.8770	-.0253	-.6505	-.0863	3
1.8	2.3809	.0641	1.8056	-.0327	-.7680	-.0608	3
1.9	2.5574	.0606	1.7251	-.0375	-.8352	-.0359	3
2.0	2.7257	.0567	1.6401	-.0399	-.8575	-.0135	3
2.1	2.8855	.0526	1.5548	-.0403	-.8425	.0054	3
2.2	3.0368	.0487	1.4725	-.0390	-.7986	.0202	3
2.3	3.1801	.0449	1.3958	-.0364	-.7343	.0307	3
2.4	3.3162	.0414	1.3261	-.0330	-.6575	.0372	3
2.5	3.4456	.0383	1.2645	-.0291	-.5747	.0403	3
2.6	3.5693	.0356	1.2112	-.0250	-.4914	.0405	3
2.7	3.6881	.0333	1.1661	-.0210	-.4116	.0387	3
2.8	3.8028	.0314	1.1286	-.0173	-.3381	.0355	3
2.9	3.9141	.0298	1.0982	-.0140	-.2726	.0314	3
3.0	4.0227	.0286	1.0738	-.0111	-.2159	.0269	3
3.1	4.1290	.0276	1.0547	-.0086	-.1680	.0225	3
3.2	4.2337	.0268	1.0399	-.0065	-.1285	.0183	3
3.3	4.3371	.0263	1.0287	-.0049	-.0967	.0146	3
3.4	4.4396	.0259	1.0204	-.0036	-.0715	.0114	3
3.5	4.5413	.0255	1.0142	-.0026	-.0520	.0087	3
3.6	4.6425	.0253	1.0098	-.0018	-.0372	.0065	3
3.7	4.7433	.0252	1.0066	-.0013	-.0263	.0048	3
3.8	4.8438	.0251	1.0044	-.0009	-.0181	.0034	3
3.9	4.9442	.0250	1.0029	-.0006	-.0124	.0024	3
4.0	5.0444	.0249	1.0019	-.0004	-.0083	.0017	3
4.1	5.1446	.0249	1.0012	-.0002	-.0055	.0012	3
4.2	5.2447	.0249	1.0008	-.0001	-.0036	.0008	3
4.3	5.3448	.0249	1.0005	-.0001	-.0023	.0005	3
4.4	5.4448	.0249	1.0003	.0000	-.0014	.0004	3
4.5	5.5448	.0249	1.0002	.0000	-.0009	.0003	3
4.6	5.6448	.0249	1.0001	.0000	-.0005	.0002	3
4.7	5.7448	.0249	1.0001	.0000	-.0003	.0001	3
4.8	5.8449	.0249	1.0001	.0000	-.0002	.0000	3
4.9	5.9449	.0249	1.0000	.0000	-.0001	.0000	3

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TABLE V. - SOLUTIONS OF FIRST-ORDER ENERGY EQUATION FOR

ARBITRARY RATES OF HEAT TRANSFER

[Pr, 0.72; γ, 1.40]

(a) The function $IH(\eta) = \int_0^\eta H\, d\eta$

η	IH_{11}	IH_{12}	IH_{13}	IH_{14}	IH_{15}	N
0	0.0000	0.0000	0.0000	0.0000	0.0000	1
.1	.0014	.0001	-.0013	.0278	.0267	1
.2	.0078	.0005	-.0027	.1111	.1068	1
.3	.0223	.0013	-.0010	.2494	.2396	1
.4	.0473	.0025	.0059	.4416	.4238	1
.5	.0848	.0042	.0192	.6859	.6571	1
.6	.1359	.0061	.0394	.9792	.9357	1
.7	.2014	.0082	.0660	1.3173	1.2549	1
.8	.2813	.0103	.0982	1.6949	1.6087	1
.9	.3751	.0120	.1347	2.1057	1.9901	1
1.0	.4818	.0134	.1739	2.5422	2.3914	1
1.1	.5999	.0141	.2141	2.9965	2.8043	1
1.2	.7277	.0141	.2535	3.4601	3.2205	1
1.3	.8631	.0134	.2907	3.9245	3.6320	1
1.4	1.0041	.0118	.3244	4.3816	4.0312	1
1.5	1.1486	.0096	.3537	4.8235	4.4117	1
1.6	1.2946	.0067	.3780	5.2437	4.7679	1
1.7	1.4167	.0034	.3970	5.6366	5.0958	1
1.8	1.5845	-.0002	.4111	5.9980	5.3924	1
1.9	1.7258	-.0041	.4206	6.3250	5.6563	1
2.0	1.8637	-.0079	.4260	6.6161	5.8874	1
2.1	1.9976	-.0116	.4281	6.8712	6.0863	1
2.2	2.1272	-.0151	.4277	7.0912	6.2550	1
2.3	2.2527	-.0183	.4255	7.2780	6.3956	1
2.4	2.3742	-.0212	.4221	7.4341	6.5111	1
2.5	2.4920	-.0237	.4181	7.5625	6.6044	1
2.6	2.6065	-.0258	.4138	7.6666	6.6787	1
2.7	2.7181	-.0276	.4097	7.7497	6.7370	1
2.8	2.8273	-.0291	.4059	7.8149	6.7820	1
2.9	2.9344	-.0303	.4025	7.8654	6.8162	1
3.0	3.0398	-.0313	.3996	7.9040	6.8418	1
3.1	3.1439	-.0320	.3972	7.9329	6.8607	1
3.2	3.2469	-.0325	.3953	7.9543	6.8745	1
3.3	3.3492	-.0330	.3938	7.9699	6.8844	1
3.4	3.4508	-.0333	.3926	7.9811	6.8913	1
3.5	3.5519	-.0335	.3917	7.9890	6.8962	1
3.6	3.6527	-.0337	.3911	7.9946	6.8995	1
3.7	3.7532	-.0338	.3906	7.9984	6.9017	1
3.8	3.8536	-.0339	.3903	8.0009	6.9032	1
3.9	3.9538	-.0339	.3901	8.0026	6.9042	1
4.0	4.0540	-.0339	.3899	8.0038	6.9049	1
4.1	4.1541	-.0340	.3898	8.0045	6.9053	1
4.2	4.2542	-.0340	.3898	8.0050	6.9055	1
4.3	4.3542	-.0340	.3897	8.0053	6.9057	1
4.4	4.4542	-.0340	.3897	8.0054	6.9058	1
4.5	4.5543	-.0340	.3897	8.0056	6.9058	1
4.6	4.6543	-.0340	.3897	8.0056	6.9059	1
4.7	4.7543	-.0340	.3897	8.0057	6.9059	1
4.8	4.8543	-.0340	.3897	8.0057	6.9059	1
4.9	4.9543	-.0340	.3897	8.0057	6.9059	1
5.0	5.0543	-.0340	.3897	8.0057	6.9059	1

TABLE V. - Continued. SOLUTIONS OF FIRST-ORDER ENERGY EQUATION FOR ARBITRARY RATES OF HEAT TRANSFER

[Pr, 0.72; γ, 1.40]

(a) Continued. The function $IH(\eta) = \int_0^\eta H\,d\eta$

η	IH_{21}	IH_{22}	IH_{23}	IH_{24}	IH_{25}	N
0	0.0000	0.0000	0.0000	0.0000	0.0000	2
.1	.0019	.0002	.0002	.0435	.0509	2
.2	.0109	.0009	.0055	.1738	.2033	2
.3	.0316	.0025	.0210	.3899	.4559	2
.4	.0672	.0051	.0499	.6899	.8055	2
.5	.1199	.0086	.0936	1.0698	1.2465	2
.6	.1910	.0128	.1520	1.5239	1.7710	2
.7	.2808	.0176	.2237	2.0449	2.3685	2
.8	.3888	.0224	.3063	2.6232	3.0266	2
.9	.5137	.0270	.3969	3.2481	3.7311	2
1.0	.6538	.0312	.4922	3.9073	4.4668	2
1.1	.8067	.0345	.5887	4.5881	5.2180	2
1.2	.9696	.0367	.6832	5.2774	5.9694	2
1.3	1.1397	.0379	.7729	5.9623	6.7064	2
1.4	1.3142	.0378	.8554	6.6308	7.4161	2
1.5	1.4903	.0366	.9290	7.2721	8.0874	2
1.6	1.6657	.0344	.9927	7.8771	8.7115	2
1.7	1.8383	.0314	1.0460	8.4384	9.2819	2
1.8	2.0063	.0278	1.0892	8.9508	9.7947	2
1.9	2.1688	.0238	1.1229	9.4112	10.2484	2
2.0	2.3249	.0196	1.1481	9.8183	10.6433	2
2.1	2.4743	.0154	1.1661	10.1727	10.9816	2
2.2	2.6169	.0113	1.1780	10.4764	11.2669	2
2.3	2.7531	.0075	1.1852	10.7328	11.5039	2
2.4	2.8833	.0041	1.1889	10.9458	11.6977	2
2.5	3.0081	.0010	1.1900	11.1202	11.8538	2
2.6	3.1282	-.0016	1.1894	11.2607	11.9776	2
2.7	3.2441	-.0038	1.1878	11.3724	12.0744	2
2.8	3.3566	-.0056	1.1857	11.4597	12.1489	2
2.9	3.4662	-.0071	1.1834	11.5270	12.2054	2
3.0	3.5735	-.0083	1.1812	11.5781	12.2477	2
3.1	3.6790	-.0092	1.1792	11.6164	12.2788	2
3.2	3.7830	-.0099	1.1775	11.6446	12.3014	2
3.3	3.8859	-.0105	1.1761	11.6650	12.3176	2
3.4	3.9880	-.0109	1.1750	11.6797	12.3290	2
3.5	4.0895	-.0111	1.1741	11.6900	12.3369	2
3.6	4.1905	-.0114	1.1734	11.6972	12.3423	2
3.7	4.2912	-.0115	1.1729	11.7021	12.3460	2
3.8	4.3917	-.0116	1.1726	11.7054	12.3484	2
3.9	4.4920	-.0117	1.1723	11.7076	12.3500	2
4.0	4.5922	-.0117	1.1722	11.7091	12.3510	2
4.1	4.6924	-.0118	1.1720	11.7100	12.3517	2
4.2	4.7924	-.0118	1.1720	11.7106	12.3521	2
4.3	4.8925	-.0118	1.1719	11.7110	12.3523	2
4.4	4.9925	-.0118	1.1719	11.7112	12.3525	2
4.5	5.0926	-.0118	1.1719	11.7114	12.3526	2
4.6	5.1926	-.0118	1.1719	11.7115	12.3526	2
4.7	5.2926	-.0118	1.1719	11.7115	12.3527	2
4.8	5.3926	-.0118	1.1719	11.7115	12.3527	2
4.9	5.4926	-.0118	1.1719	11.7116	12.3527	2
5.0	5.5926	-.0118	1.1719	11.7116	12.3527	2
5.1	5.6926	-.0118	1.1719	11.7116	12.3527	2
5.2	5.7926	-.0118	1.1719	11.7116	12.3527	2

TABLE V. - Continued. SOLUTIONS OF FIRST-ORDER ENERGY EQUATION FOR ARBITRARY RATES OF HEAT TRANSFER

[Pr, 0.72; γ, 1.40]

(a) Concluded. The function $IH(\eta) = \int_0^\eta H\,d\eta$

η	IH_{31}	IH_{32}	IH_{33}	IH_{34}	IH_{35}	N
0	0.0000	0.0000	0.0000	0.0000	0.0000	3
.1	.0023	.0002	.0023	.0569	.0727	3
.2	.0135	.0013	.0153	.2274	.2905	3
.3	.0391	.0037	.0458	.5101	.6510	3
.4	.0830	.0074	.0977	.9015	1.1488	3
.5	.1475	.0126	.1720	1.3958	1.7749	3
.6	.2338	.0189	.2680	1.9848	2.5165	3
.7	.3417	.0259	.3830	2.6573	3.3574	3
.8	.4703	.0333	.5132	3.4002	4.2786	3
.9	.6178	.0406	.6541	4.1984	5.2591	3
1.0	.7815	.0473	.8010	5.0358	6.2769	3
1.1	.9586	.0530	.9490	5.8954	7.3100	3
1.2	1.1458	.0576	1.0936	6.7605	8.3370	3
1.3	1.3397	.0607	1.2309	7.6150	9.3387	3
1.4	1.5369	.0625	1.3577	8.4443	10.2979	3
1.5	1.7345	.0628	1.4719	9.2354	11.2006	3
1.6	1.9296	.0618	1.5720	9.9778	12.0357	3
1.7	2.1201	.0597	1.6575	10.6630	12.7956	3
1.8	2.3041	.0568	1.7285	11.2857	13.4761	3
1.9	2.4806	.0532	1.7860	11.8426	14.0758	3
2.0	2.6488	.0493	1.8311	12.3330	14.5961	3
2.1	2.8085	.0453	1.8655	12.7582	15.0405	3
2.2	2.9598	.0413	1.8907	13.1214	15.4144	3
2.3	3.1032	.0375	1.9086	13.4268	15.7242	3
2.4	3.2392	.0341	1.9207	13.6798	15.9770	3
2.5	3.3686	.0310	1.9284	13.8863	16.1803	3
2.6	3.4923	.0282	1.9329	14.0523	16.3412	3
2.7	3.6111	.0259	1.9352	14.1837	16.4669	3
2.8	3.7258	.0240	1.9360	14.2863	16.5635	3
2.9	3.8371	.0225	1.9359	14.3652	16.6367	3
3.0	3.9456	.0212	1.9353	14.4250	16.6914	3
3.1	4.0520	.0202	1.9344	14.4697	16.7316	3
3.2	4.1567	.0195	1.9336	14.5025	16.7608	3
3.3	4.2601	.0189	1.9328	14.5263	16.7816	3
3.4	4.3626	.0185	1.9321	14.5433	16.7963	3
3.5	4.4643	.0182	1.9315	14.5553	16.8066	3
3.6	4.5655	.0180	1.9311	14.5636	16.8135	3
3.7	4.6663	.0178	1.9308	14.5693	16.8182	3
3.8	4.7668	.0177	1.9305	14.5731	16.8214	3
3.9	4.8672	.0176	1.9304	14.5756	16.8234	3
4.0	4.9674	.0175	1.9303	14.5773	16.8247	3
4.1	5.0676	.0175	1.9302	14.5784	16.8256	3
4.2	5.1677	.0175	1.9301	14.5790	16.8261	3
4.3	5.2677	.0175	1.9301	14.5795	16.8264	3
4.4	5.3678	.0174	1.9301	14.5797	16.8266	3
4.5	5.4678	.0174	1.9301	14.5799	16.8268	3
4.6	5.5678	.0174	1.9301	14.5800	16.8268	3
4.7	5.6678	.0174	1.9301	14.5801	16.8269	3
4.8	5.7678	.0174	1.9301	14.5801	16.8269	3
4.9	5.8678	.0174	1.9301	14.5801	16.8269	3
5.0	5.9678	.0174	1.9301	14.5801	16.8269	3
5.1	6.0678	.0174	1.9301	14.5801	16.8269	3
5.2	6.1678	.0174	1.9301	14.5801	16.8269	3
5.3	6.2678	.0174	1.9301	14.5801	16.8269	3

TABLE V. - Continued. SOLUTIONS OF FIRST-ORDER ENERGY EQUATION FOR ARBITRARY RATES OF HEAT TRANSFER

[Pr, 0.72; γ, 1.40]

(b) The function $H(\eta)$

η	H_{11}	H_{12}	H_{13}	H_{14}	H_{15}	N
0	0.0000	0.0000	0.0000	0.0000	0.0000	1
.1	.0333	.0020	-.0198	.5556	.5343	1
.2	.1000	.0057	-.0028	1.1091	1.0659	1
.3	.1942	.0102	.0404	1.6554	1.5884	1
.4	.3100	.0147	.1000	2.1869	2.0923	1
.5	.4414	.0183	.1674	2.6938	2.5664	1
.6	.5825	.0206	.2350	3.1648	2.9983	1
.7	.7272	.0210	.2962	3.5885	3.3760	1
.8	.8696	.0194	.3462	3.9534	3.6884	1
.9	1.0044	.0159	.3813	4.2493	3.9267	1
1.0	1.1268	.0106	.3997	4.4679	4.0846	1
1.1	1.2328	.0039	.4008	4.6038	4.1593	1
1.2	1.3197	-.0037	.3856	4.6543	4.1516	1
1.3	1.3857	-.0115	.3563	4.6205	4.0655	1
1.4	1.4306	-.0190	.3160	4.5068	3.9088	1
1.5	1.4552	-.0257	.2682	4.3211	3.6916	1
1.6	1.4613	-.0312	.2168	4.0738	3.4262	1
1.7	1.4516	-.0351	.1653	3.7773	3.1259	1
1.8	1.4291	-.0376	.1167	3.4454	2.8043	1
1.9	1.3973	-.0384	.0733	3.0920	2.4743	1
2.0	1.3592	-.0379	.0367	2.7305	2.1478	1
2.1	1.3178	-.0362	.0075	2.3731	1.8344	1
2.2	1.2755	-.0336	-.0142	2.0301	1.5419	1
2.3	1.2344	-.0305	-.0291	1.7096	1.2756	1
2.4	1.1958	-.0269	-.0380	1.4174	1.0390	1
2.5	1.1608	-.0233	-.0420	1.1570	.8332	1
2.6	1.1299	-.0197	-.0423	.9301	.6580	1
2.7	1.1032	-.0163	-.0401	.7363	.5117	1
2.8	1.0807	-.0133	-.0362	.5740	.3920	1
2.9	1.0621	-.0106	-.0314	.4408	.2958	1
3.0	1.0471	-.0083	-.0264	.3335	.2199	1
3.1	1.0352	-.0064	-.0216	.2485	.1611	1
3.2	1.0259	-.0049	-.0172	.1824	.1163	1
3.3	1.0188	-.0036	-.0134	.1320	.0827	1
3.4	1.0134	-.0026	-.0101	.0940	.0579	1
3.5	1.0094	-.0019	-.0075	.0660	.0400	1
3.6	1.0066	-.0013	-.0055	.0457	.0272	1
3.7	1.0045	-.0009	-.0039	.0311	.0183	1
3.8	1.0030	-.0006	-.0027	.0209	.0121	1
3.9	1.0020	-.0004	-.0018	.0138	.0079	1
4.0	1.0013	-.0003	-.0012	.0090	.0051	1
4.1	1.0008	-.0002	-.0007	.0058	.0032	1
4.2	1.0005	-.0001	-.0004	.0037	.0020	1
4.3	1.0003	-.0001	-.0002	.0023	.0012	1
4.4	1.0002	.0000	-.0001	.0014	.0007	1
4.5	1.0001	.0000	.0000	.0009	.0004	1
4.6	1.0001	.0000	.0000	.0005	.0003	1
4.7	1.0001	.0000	.0000	.0003	.0001	1
4.8	1.0000	.0000	.0000	.0002	.0001	1
4.9	1.0000	.0000	.0000	.0001	.0000	1
5.0	1.0000	.0000	.0000	.0000	.0000	1

TABLE V. - Continued. SOLUTIONS OF FIRST-ORDER ENERGY EQUATION FOR ARBITRARY RATES OF HEAT TRANSFER

[Pr, 0.72; γ, 1.40]

(b) Continued. The function $H(\eta)$

η	H_{21}	H_{22}	H_{23}	H_{24}	H_{25}	N
0	0.0000	0.0000	0.0000	0.0000	0.0000	2
.1	.0465	.0039	.0170	.8695	1.0177	2
.2	.1424	.0115	.0969	1.7347	2.0287	2
.3	.2766	.0210	.2183	2.5854	3.0179	2
.4	.4384	.0306	.3622	3.4069	3.9641	2
.5	.6176	.0391	.5118	4.1813	4.8429	2
.6	.8046	.0452	.6534	4.8898	5.6294	2
.7	.9905	.0483	.7760	5.5134	6.3007	2
.8	1.1676	.0480	.8716	6.0353	6.8378	2
.9	1.3288	.0443	.9352	6.4415	7.2269	2
1.0	1.4688	.0375	.9646	6.7222	7.4606	2
1.1	1.5833	.0281	.9605	6.8722	7.5379	2
1.2	1.6699	.0171	.9255	6.8917	7.4650	2
1.3	1.7278	.0054	.8642	6.7862	7.2538	2
1.4	1.7575	-.0063	.7827	6.5658	6.9215	2
1.5	1.7614	-.0170	.6874	6.2451	6.4894	2
1.6	1.7427	-.0263	.5851	5.8418	5.9810	2
1.7	1.7055	-.0335	.4819	5.3758	5.4210	2
1.8	1.6543	-.0386	.3831	4.8677	4.8333	2
1.9	1.5936	-.0415	.2928	4.3379	4.2402	2
2.0	1.5276	-.0424	.2137	3.8052	3.6610	2
2.1	1.4533	-.0416	.1472	3.2861	3.1114	2
2.2	1.3937	-.0394	.0937	2.7941	2.6033	2
2.3	1.3312	-.0362	.0524	2.3394	2.1448	2
2.4	1.2740	-.0324	.0222	1.9290	1.7402	2
2.5	1.2230	-.0283	.0014	1.5665	1.3907	2
2.6	1.1787	-.0242	-.0118	1.2531	1.0947	2
2.7	1.1409	-.0202	-.0192	.9874	.8489	2
2.8	1.1095	-.0165	-.0224	.7664	.6486	2
2.9	1.0838	-.0133	-.0226	.5861	.4882	2
3.0	1.0631	-.0105	-.0211	.4416	.3621	2
3.1	1.0469	-.0081	-.0185	.3278	.2647	2
3.2	1.0343	-.0062	-.0156	.2398	.1907	2
3.3	1.0247	-.0046	-.0127	.1729	.1354	2
3.4	1.0176	-.0034	-.0100	.1228	.0947	2
3.5	1.0123	-.0025	-.0076	.0859	.0653	2
3.6	1.0085	-.0017	-.0057	.0592	.0444	2
3.7	1.0058	-.0012	-.0042	.0402	.0297	2
3.8	1.0039	-.0009	-.0030	.0269	.0196	2
3.9	1.0026	-.0006	-.0021	.0178	.0128	2
4.0	1.0017	-.0004	-.0014	.0116	.0082	2
4.1	1.0011	-.0003	-.0010	.0074	.0052	2
4.2	1.0007	-.0002	-.0006	.0047	.0032	2
4.3	1.0004	-.0001	-.0004	.0029	.0020	2
4.4	1.0003	-.0001	-.0003	.0018	.0012	2
4.5	1.0002	-.0001	-.0002	.0011	.0007	2
4.6	1.0001	-.0001	-.0002	.0007	.0004	2
4.7	1.0001	.0000	-.0001	.0004	.0002	2
4.8	1.0001	.0000	-.0001	.0002	.0001	2
4.9	1.0000	.0000	-.0001	.0001	.0001	2
5.0	1.0000	.0000	.0000	.0001	.0001	2
5.1	1.0000	.0000	.0000	.0001	.0001	2
5.2	1.0000	.0000	.0000	.0001	.0001	2
5.3	1.0000	.0000	.0000	.0000	.0000	2

TABLE V. - Continued. SOLUTIONS OF FIRST-ORDER ENERGY EQUATION FOR ARBITRARY RATES OF HEAT TRANSFER

[Pr, 0.72; γ, 1.40]

(b) Concluded. The function $H(\eta)$

η	H_{31}	H_{32}	H_{33}	H_{34}	H_{35}	N
0	0.0000	0.0000	0.0000	0.0000	0.0000	3
.1	.0473	.0056	.0624	1.1383	1.4545	3
.2	.1767	.0167	.2090	2.2695	2.8973	3
.3	.3423	.0306	.4079	3.3778	4.3033	3
.4	.5390	.0449	.6310	4.4407	5.6378	3
.5	.7529	.0576	.8547	5.4327	6.8629	3
.6	.9719	.0673	1.0601	6.3275	7.9423	3
.7	1.1853	.0729	1.2330	7.1008	8.8443	3
.8	1.3840	.0739	1.3638	7.7320	9.5447	3
.9	1.5607	.0705	1.4473	8.2059	10.0285	3
1.0	1.7097	.0629	1.4819	8.5131	10.2903	3
1.1	1.8271	.0520	1.4697	8.6511	10.3346	3
1.2	1.9109	.0388	1.4153	8.6240	10.1744	3
1.3	1.9609	.0244	1.3255	8.4422	9.8307	3
1.4	1.9788	.0099	1.2084	8.1217	9.3302	3
1.5	1.9675	-.0037	1.0730	7.6829	8.7040	3
1.6	1.9313	-.0157	.9280	7.1496	7.9850	3
1.7	1.8751	-.0255	.7816	6.5470	7.2067	3
1.8	1.8043	-.0328	.6406	5.9009	6.4007	3
1.9	1.7242	-.0376	.5104	5.2359	5.5957	3
2.0	1.6395	-.0400	.3947	4.5744	4.8162	3
2.1	1.5544	-.0403	.2954	3.9355	4.0818	3
2.2	1.4723	-.0390	.2132	3.3344	3.4068	3
2.3	1.3956	-.0364	.1475	2.7827	2.8007	3
2.4	1.3260	-.0330	.0968	2.2874	2.2679	3
2.5	1.2644	-.0291	.0591	1.8523	1.8092	3
2.6	1.2111	-.0250	.0323	1.4776	1.4220	3
2.7	1.1660	-.0211	.0141	1.1614	1.1012	3
2.8	1.1286	-.0173	.0026	.8994	.8403	3
2.9	1.0982	-.0140	-.0041	.6862	.6319	3
3.0	1.0738	-.0111	-.0075	.5160	.4683	3
3.1	1.0547	-.0086	-.0086	.3823	.3420	3
3.2	1.0399	-.0066	-.0084	.2791	.2462	3
3.3	1.0287	-.0049	-.0075	.2008	.1747	3
3.4	1.0204	-.0036	-.0062	.1424	.1221	3
3.5	1.0142	-.0026	-.0050	.0994	.0841	3
3.6	1.0098	-.0019	-.0038	.0684	.0571	3
3.7	1.0066	-.0013	-.0028	.0464	.0382	3
3.8	1.0044	-.0009	-.0020	.0310	.0252	3
3.9	1.0029	-.0006	-.0014	.0204	.0164	3
4.0	1.0019	-.0005	-.0009	.0133	.0105	3
4.1	1.0012	-.0003	-.0006	.0085	.0066	3
4.2	1.0008	-.0002	-.0004	.0054	.0041	3
4.3	1.0005	-.0002	-.0002	.0033	.0025	3
4.4	1.0003	-.0002	-.0001	.0021	.0015	3
4.5	1.0002	-.0001	.0000	.0012	.0009	3
4.6	1.0001	.0000	.0000	.0007	.0005	3
4.7	1.0001	.0000	.0000	.0004	.0003	3
4.8	1.0001	.0000	.0000	.0003	.0002	3
4.9	1.0000	.0000	.0000	.0002	.0001	3
5.0	1.0000	.0000	.0000	.0001	.0000	3
5.1	1.0000	.0000	.0000	.0001	.0000	3
5.2	1.0000	.0000	.0000	.0001	.0000	3
5.3	1.0000	.0000	.0000	.0000	.0000	3

TABLE V. - Continued. SOLUTIONS OF FIRST-ORDER ENERGY EQUATION FOR ARBITRARY RATES OF HEAT TRANSFER

[Pr, 0.72; γ, 1.40]

(c) The function H'(η)

η	H'_{11}	H'_{12}	H'_{13}	H'_{14}	H'_{15}	N
0	0.1479	0.0082	-0.4201	5.5574	5.3452	1
.1	.5096	.0302	.0042	5.5512	5.3370	1
.2	.8138	.0427	.3179	5.5101	5.2837	1
.3	1.0598	.0462	.5298	5.4040	5.1497	1
.4	1.2464	.0416	.6492	5.2093	4.9101	1
.5	1.3728	.0303	.6865	4.9087	4.5506	1
.6	1.4386	.0139	.6534	4.4931	4.0675	1
.7	1.4450	-.0055	.5632	3.9610	3.4675	1
.8	1.3949	-.0259	.4301	3.3197	2.7665	1
.9	1.2932	-.0450	.2695	2.5847	1.9888	1
1.0	1.1477	-.0609	.0968	1.7794	1.1652	1
1.1	.9682	-.0722	-.0731	.9329	.3306	1
1.2	.7662	-.0778	-.2268	.0789	-.4790	1
1.3	.5543	-.0775	-.3538	-.7475	-1.2289	1
1.4	.3450	-.0717	-.4467	-1.5121	-1.8888	1
1.5	.1495	-.0613	-.5020	-2.1847	-2.4349	1
1.6	-.0230	-.0476	-.5203	-2.7412	-2.8515	1
1.7	-.1663	-.0321	-.5051	-3.1653	-3.1320	1
1.8	-.2771	-.0163	-.4630	-3.4497	-3.2784	1
1.9	-.3550	-.0014	-.4018	-3.5959	-3.3005	1
2.0	-.4020	.0115	-.3296	-3.6132	-3.2145	1
2.1	-.4220	.0218	-.2542	-3.5176	-3.0405	1
2.2	-.4196	.0293	-.1817	-3.3294	-2.8009	1
2.3	-.4003	.0341	-.1169	-3.0715	-2.5181	1
2.4	-.3690	.0363	-.0625	-2.7669	-2.2129	1
2.5	-.3303	.0364	-.0198	-2.4374	-1.9033	1
2.6	-.2882	.0348	.0114	-2.1021	-1.6038	1
2.7	-.2456	.0321	.0323	-1.7765	-1.3251	1
2.8	-.2048	.0287	.0444	-1.4722	-1.0743	1
2.9	-.1673	.0248	.0496	-1.1972	-.8550	1
3.0	-.1340	.0210	.0497	-.9558	-.6684	1
3.1	-.1054	.0173	.0465	-.7495	-.5135	1
3.2	-.0814	.0139	.0412	-.5775	-.3878	1
3.3	-.0617	.0110	.0353	-.4373	-.2879	1
3.4	-.0458	.0085	.0291	-.3256	-.2102	1
3.5	-.0338	.0064	.0233	-.2384	-.1511	1
3.6	-.0241	.0048	.0181	-.1716	-.1067	1
3.7	-.0175	.0035	.0137	-.1217	-.0743	1
3.8	-.0122	.0025	.0102	-.0844	-.0506	1
3.9	-.0083	.0018	.0074	-.0580	-.0342	1
4.0	-.0056	.0012	.0053	-.0392	-.0227	1
4.1	-.0037	.0008	.0037	-.0261	-.0149	1
4.2	-.0024	.0006	.0025	-.0169	-.0095	1
4.3	-.0015	.0004	.0017	-.0110	-.0061	1
4.4	-.0010	.0002	.0012	-.0069	-.0038	1
4.5	-.0006	.0002	.0008	-.0044	-.0023	1
4.6	-.0004	.0001	.0005	-.0026	-.0014	1
4.7	-.0002	.0001	.0003	-.0016	-.0009	1
4.8	-.0001	.0000	.0001	-.0009	-.0005	1
4.9	.0000	.0000	.0001	-.0005	-.0003	1
5.0	.0000	.0000	.0000	-.0003	-.0002	1
5.1	.0000	.0000	.0000	-.0002	-.0001	1
5.2	.0000	.0000	.0000	-.0001	.0000	1
5.3	.0000	.0000	.0000	.0000	.0000	1

TABLE V. - Continued. SOLUTIONS OF FIRST-ORDER ENERGY EQUATION FOR ARBITRARY RATES OF HEAT TRANSFER

[Pr, 0.72; γ, 1.40]

(c) Continued. The function $H'(\eta)$

η	H'_{21}	H'_{22}	H'_{23}	H'_{24}	H'_{25}	N
0	0.1802	0.0145	-0.2200	8.6985	10.1820	2
.1	.7308	.0608	.5214	8.6855	10.1618	2
.2	1.1688	.0882	1.0403	8.6008	10.0325	2
.3	1.4974	.0981	1.3564	8.3886	9.7161	2
.4	1.7214	.0926	1.4932	8.0108	9.2336	2
.5	1.8462	.0745	1.4771	7.4464	8.3668	2
.6	1.8787	.0472	1.3366	6.6905	7.3248	2
.7	1.8273	.0143	1.1015	5.7539	6.0700	2
.8	1.7019	-.0205	.8017	4.6607	4.6490	2
.9	1.5142	-.0536	.4665	3.4470	3.1207	2
1.0	1.2780	-.0819	.1237	2.1577	1.5507	2
1.1	1.0086	-.1030	-.2021	.8433	.0071	2
1.2	.7223	-.1154	-.4901	-.4432	-1.4456	2
1.3	.4356	-.1186	-.7244	-1.6506	-2.7499	2
1.4	.1637	-.1133	-.8953	-2.7328	-3.8598	2
1.5	-.0803	-.1008	-.9989	-3.6519	-4.7432	2
1.6	-.2868	-.0829	-1.0374	-4.3809	-5.3830	2
1.7	-.4498	-.0618	-1.0179	-4.9051	-5.7775	2
1.8	-.5671	-.0398	-.9513	-5.2224	-5.9390	2
1.9	-.6404	-.0187	-.8506	-5.3422	-5.8915	2
2.0	-.6738	.0001	-.7292	-5.2842	-5.6674	2
2.1	-.6735	.0157	-.5996	-5.0754	-5.3047	2
2.2	-.6467	.0275	-.4722	-4.7476	-4.8427	2
2.3	-.6007	.0355	-.3547	-4.3343	-4.3198	2
2.4	-.5421	.0401	-.2522	-3.8683	-3.7703	2
2.5	-.4770	.0417	-.1673	-3.3791	-3.2234	2
2.6	-.4102	.0409	-.1003	-2.8920	-2.7019	2
2.7	-.3453	.0383	-.0503	-2.4271	-2.2219	2
2.8	-.2848	.0347	-.0150	-1.9985	-1.7939	2
2.9	-.2305	.0304	.0081	-1.6157	-1.4225	2
3.0	-.1832	.0258	.0216	-1.2829	-1.1084	2
3.1	-.1430	.0215	.0282	-1.0009	-.8490	2
3.2	-.1097	.0174	.0298	-.7675	-.6394	2
3.3	-.0827	.0138	.0284	-.5787	-.4737	2
3.4	-.0613	.0107	.0253	-.4292	-.3452	2
3.5	-.0448	.0081	.0214	-.3131	-.2476	2
3.6	-.0321	.0060	.0174	-.2245	-.1746	2
3.7	-.0227	.0044	.0136	-.1587	-.1214	2
3.8	-.0157	.0031	.0103	-.1096	-.0826	2
3.9	-.0107	.0022	.0076	-.0750	-.0556	2
4.0	-.0072	.0015	.0056	-.0506	-.0369	2
4.1	-.0048	.0010	.0039	-.0336	-.0242	2
4.2	-.0031	.0007	.0027	-.0217	-.0154	2
4.3	-.0020	.0004	.0018	-.0140	-.0098	2
4.4	-.0012	.0002	.0012	-.0088	-.0061	2
4.5	-.0008	.0001	.0008	-.0055	-.0038	2
4.6	-.0005	.0001	.0005	-.0034	-.0023	2
4.7	-.0003	.0000	.0003	-.0021	-.0014	2
4.8	-.0002	.0000	.0001	-.0012	-.0008	2
4.9	-.0001	.0000	.0000	-.0007	-.0004	2
5.0	.0000	.0000	.0000	-.0003	-.0002	2
5.1	.0000	.0000	.0000	-.0002	-.0001	2
5.2	.0000	.0000	.0000	-.0001	-.0001	2
5.3	.0000	.0000	.0000	.0000	.0000	2

TABLE V. - Continued. SOLUTIONS OF FIRST-ORDER ENERGY EQUATION FOR ARBITRARY RATES OF HEAT TRANSFER

[Pr, 0.72; γ, 1.40]

(c) Concluded. The function $H'(\eta)$

η	H'_{31}	H'_{32}	H'_{33}	H'_{34}	H'_{35}	N
0	0.2042	0.0195	0.0899	11.3879	14.5535	3
.1	.9113	.0884	1.1008	11.3667	14.5183	3
.2	1.4518	.1292	1.7774	11.2306	14.2961	3
.3	1.8353	.1446	2.1532	10.8975	13.7641	3
.4	2.0745	.1380	2.2698	10.3188	12.8617	3
.5	2.1837	.1139	2.1731	9.4767	11.5807	3
.6	2.1783	.0774	1.9111	8.3784	9.9538	3
.7	2.0743	.0336	1.5308	7.0529	8.0448	3
.8	1.8879	-.0125	1.0768	5.5460	5.9380	3
.9	1.6362	-.0562	.5900	3.9150	3.7290	3
1.0	1.3367	-.0937	.1064	2.2247	1.5156	3
1.1	1.0079	-.1222	-.3431	.5423	-.6085	3
1.2	.6682	-.1399	-.7340	-1.0666	-2.5599	3
1.3	.3357	-.1463	-1.0489	-2.5421	-4.2699	3
1.4	.0267	-.1422	-1.2773	-3.8336	-5.6878	3
1.5	-.2454	-.1292	-1.4162	-4.9022	-6.7821	3
1.6	-.4707	-.1095	-1.4697	-5.7230	-7.5412	3
1.7	-.6437	-.0857	-1.4474	-6.2858	-7.9725	3
1.8	-.7632	-.0603	-1.3635	-6.5948	-8.0995	3
1.9	-.8318	-.0356	-1.2344	-6.6674	-7.9594	3
2.0	-.8552	-.0132	-1.0770	-6.5313	-7.5980	3
2.1	-.8409	.0056	-.9070	-6.2221	-7.0666	3
2.2	-.7976	.0203	-.7377	-5.7794	-6.4171	3
2.3	-.7336	.0307	-.5792	-5.2441	-5.6987	3
2.4	-.6570	.0375	-.4383	-4.6551	-4.9552	3
2.5	-.5743	.0403	-.3187	-4.0470	-4.2229	3
2.6	-.4911	.0405	-.2214	-3.4489	-3.5301	3
2.7	-.4115	.0387	-.1454	-2.8833	-2.8963	3
2.8	-.3380	.0355	-.0887	-2.3660	-2.3337	3
2.9	-.2726	.0314	-.0481	-1.9066	-1.8474	3
3.0	-.2159	.0269	-.0208	-1.5094	-1.4374	3
3.1	-.1680	.0225	-.0034	-1.1745	-1.0996	3
3.2	-.1285	.0183	.0065	-.8984	-.8272	3
3.3	-.0967	.0146	.0114	-.6759	-.6123	3
3.4	-.0716	.0114	.0130	-.5003	-.4460	3
3.5	-.0422	.0086	.0124	-.3642	-.3196	3
3.6	-.0373	.0064	.0109	-.2607	-.2253	3
3.7	-.0263	.0047	.0090	-.1839	-.1566	3
3.8	-.0182	.0033	.0070	-.1267	-.1064	3
3.9	-.0124	.0023	.0053	-.0865	-.0715	3
4.0	-.0083	.0016	.0039	-.0583	-.0476	3
4.1	-.0055	.0011	.0028	-.0387	-.0311	3
4.2	-.0035	.0007	.0019	-.0249	-.0198	3
4.3	-.0023	.0004	.0013	-.0160	-.0125	3
4.4	-.0014	.0002	.0009	-.0101	-.0078	3
4.5	-.0009	.0002	.0006	-.0063	-.0048	3
4.6	-.0005	.0002	.0004	-.0038	-.0029	3
4.7	-.0003	.0001	.0002	-.0023	-.0018	3
4.8	-.0002	.0001	.0001	-.0013	-.0010	3
4.9	-.0001	.0000	.0000	-.0009	-.0006	3
5.0	.0000	.0000	.0000	-.0006	-.0003	3
5.1	.0000	.0000	.0000	-.0004	-.0001	3
5.2	.0000	.0000	.0000	-.0002	.0000	3
5.3	.0000	.0000	.0000	.0001	.0000	3

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TABLE VI. - ASYMPTOTIC VALUES

APPEARING IN EXPRESSION FOR

DISPLACEMENT THICKNESS

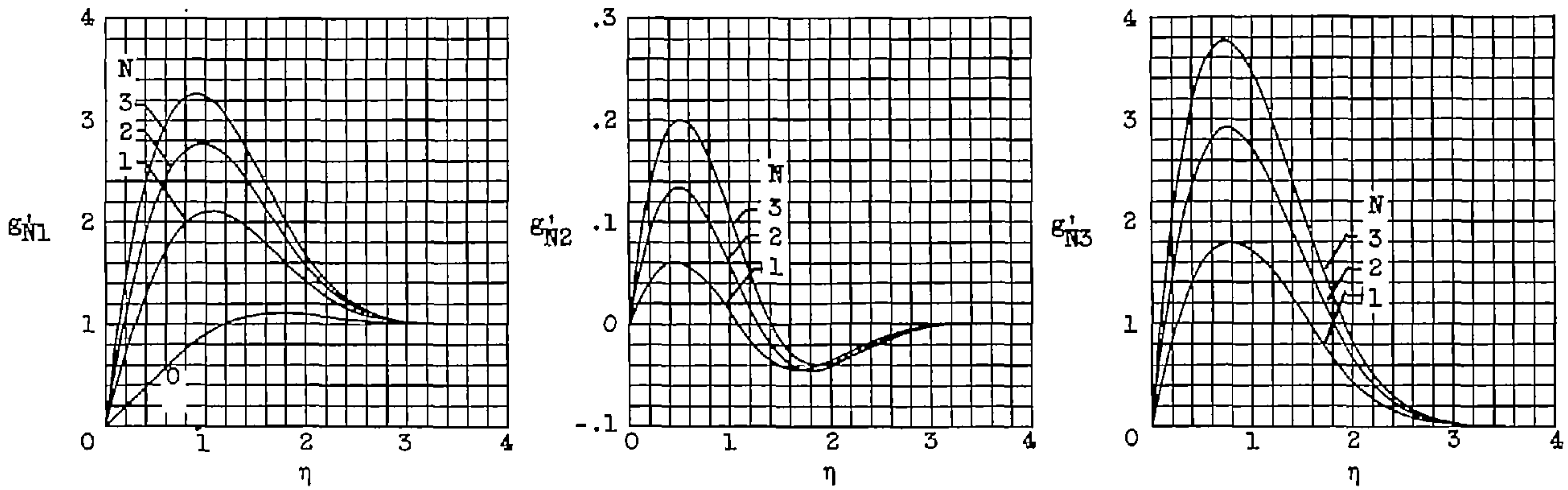
[Pr, 0.72; γ, 1.4]

	N = 1	N = 2	N = 3
α_{N1}	-4.4764	-6.5114	-7.9898
α_{N2}	.0126	-.1230	-.2688
α_{N3}	-5.1946	-8.3220	-10.6824
β_{N1}	-2.0346	-3.0786	-3.8104
β_{N2}	.0566	.0096	-.0476
B_{N1}	-1.8294	-2.9060	-3.6564
B_{N2}	.0680	.0236	-.0348
B_{N3}	-.7794	-2.3438	-3.8602
B_{N4}	-16.0112	-23.4228	-29.1598
B_{N5}	-13.8116	-24.7052	-33.6536

TABLE VII. - SOLUTION OF FIRST ORDER MOMENTUM EQUATION

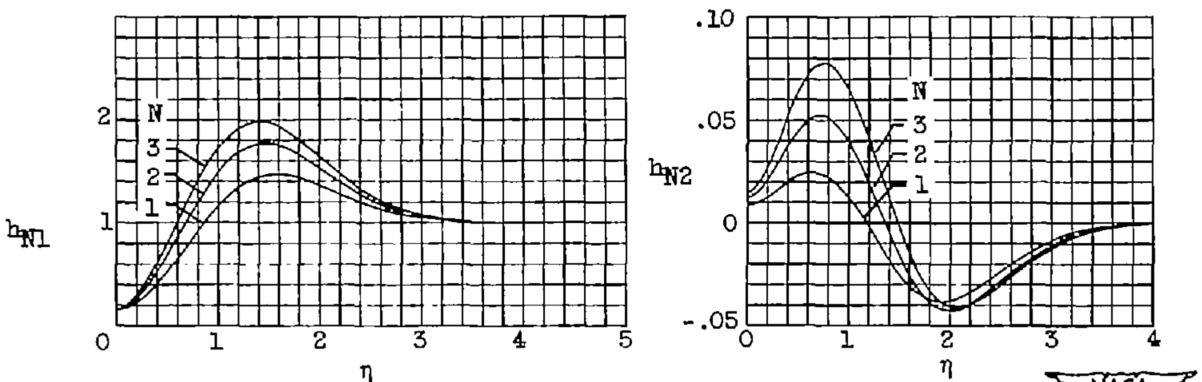
[Pr, 1; γ, 1.40]

η	g_{12}	g'_{12}	g''_{12}	η	g_{12}	g'_{12}	g''_{12}	N
0	0.0000	0.0000	0.3516	2.5	0.0222	-0.0150	0.0374	1
.1	.0016	.0312	.2719	2.6	.0209	-.0115	.0318	1
.2	.0060	.0544	.1941	2.7	.0199	-.0087	.0260	1
.3	.0123	.0701	.1203	2.8	.0191	-.0063	.0206	1
.4	.0198	.0787	.0521	2.9	.0186	-.0045	.0158	1
.5	.0278	.0808	-.0085	3.0	.0182	-.0032	.0117	1
.6	.0357	.0775	-.0599	3.1	.0180	-.0022	.0085	1
.7	.0431	.0692	-.1010	3.2	.0178	-.0014	.0060	1
.8	.0494	.0575	-.1308	3.3	.0177	-.0009	.0041	1
.9	.0545	.0434	-.1489	3.4	.0176	-.0006	.0027	1
1.0	.0581	.0281	-.1555	3.5	.0175	-.0004	.0018	1
1.1	.0601	.0126	-.1516	3.6	.0175	-.0002	.0011	1
1.2	.0606	-.0019	-.1385	3.7	.0175	-.0001	.0007	1
1.3	.0598	-.0143	-.1183	3.8	.0175	-.0001	.0004	1
1.4	.0578	-.0254	-.0933	3.9	.0175	.0000	.0003	1
1.5	.0548	-.0334	-.0660	4.0	.0175	.0000	.0002	1
1.6	.0512	-.0386	-.0387	4.1	.0175	.0000	.0001	1
1.7	.0472	-.0412	-.0135	4.2	.0175	.0000	.0000	1
1.8	.0430	-.0414	.0080	4.3	.0175	.0000	.0000	1
1.9	.0389	-.0398	.0249	4.4	.0175	.0000	.0000	1
2.0	.0351	-.0366	.0367	4.5	.0175	.0000	.0000	1
2.1	.0316	-.0326	.0437					1
2.2	.0286	-.0280	.0464					1
2.3	.0260	-.0234	.0456					1
2.4	.0239	-.0190	.0423					1



(a) g'_{N1}. (b) g'_{N2}. Pr, 0.72; γ, 1.40. (c) g'_{N3}. Pr, 0.72; γ, 1.40.

Figure 1. - Variation of functions g'_{Ni} with characteristic variable η.



(a) h_{N1}. (b) h_{N2}.

Figure 2. - Variation of functions h_{Ni} with characteristic variable η. Pr, 0.72; γ, 1.40.

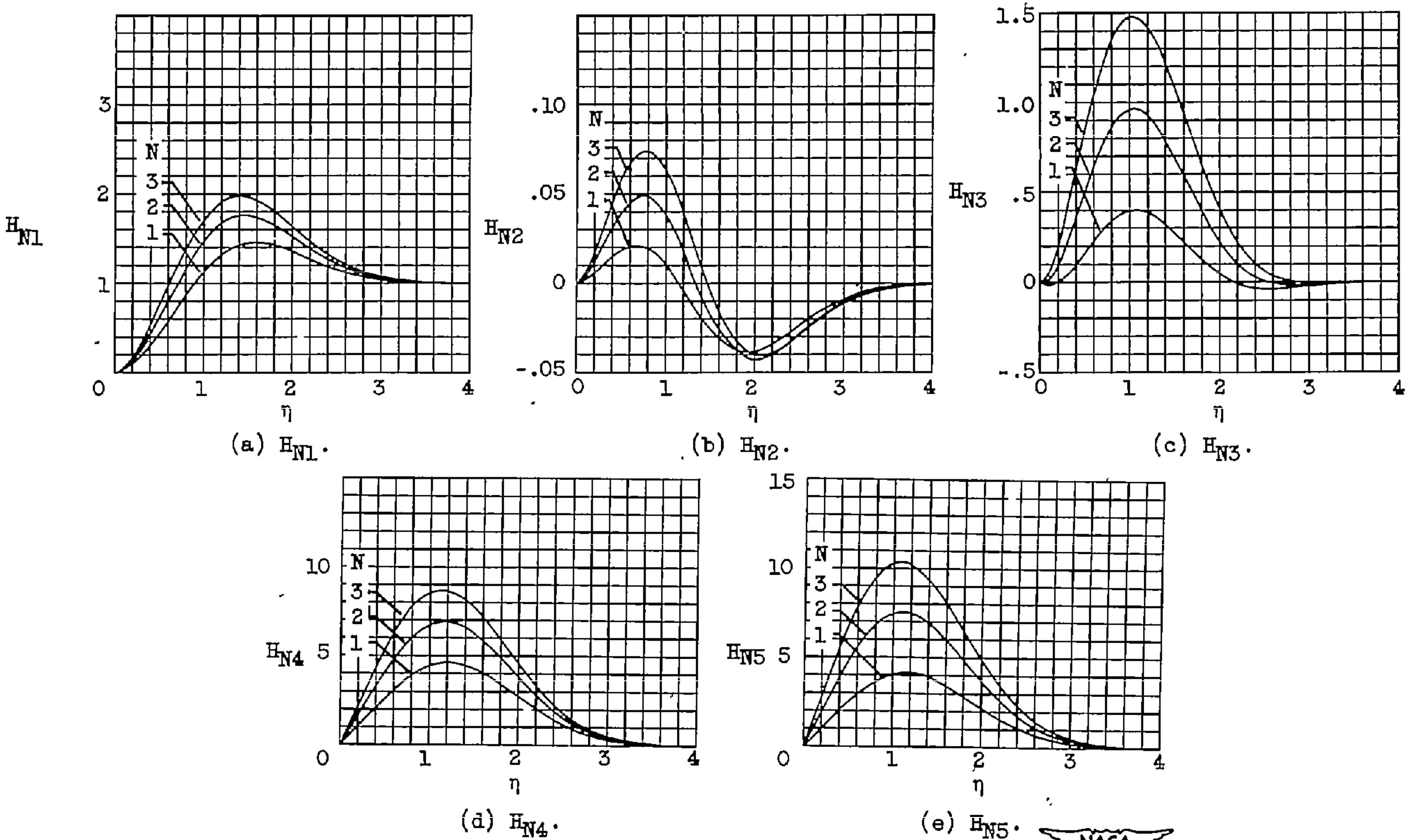


Figure 3. - Variation of functions H_{Ni} with characteristic variable η. Pr, 0.72; γ, 1.40.

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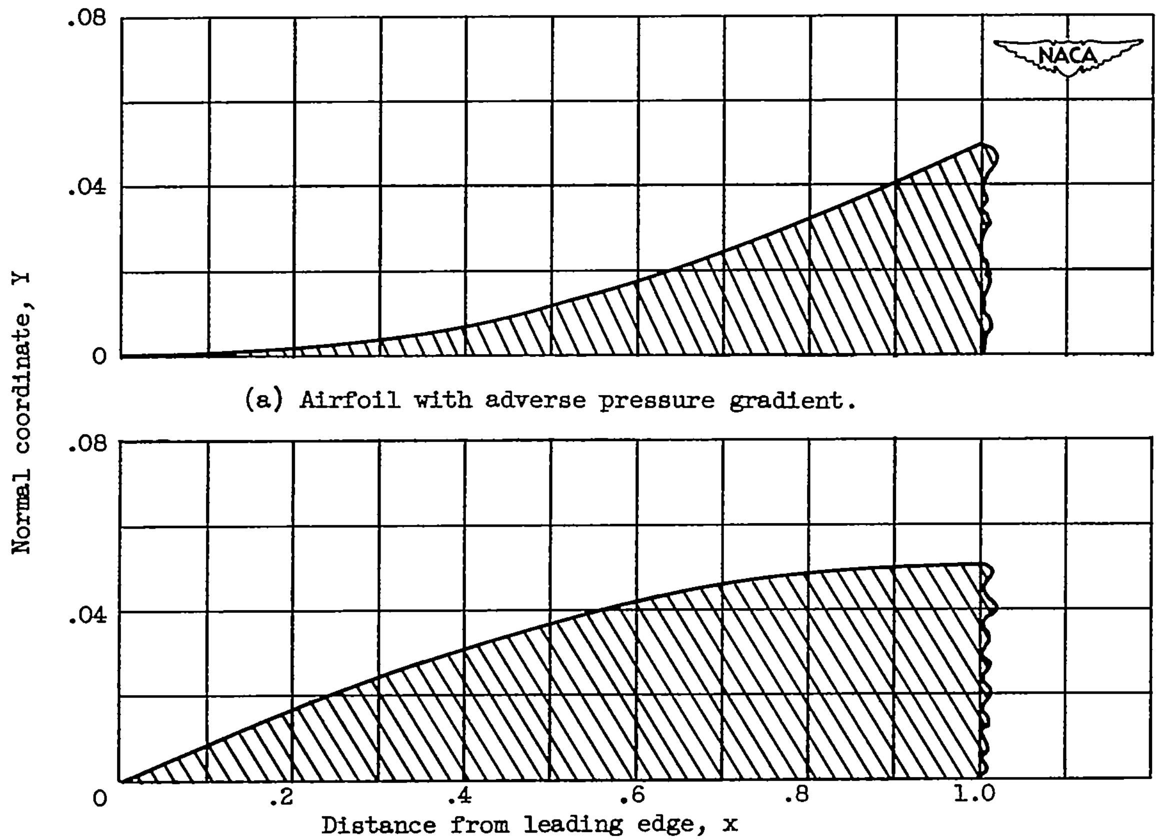

(b) Airfoil with favorable pressure gradient.

Figure 4. - Airfoil shapes used in examples.

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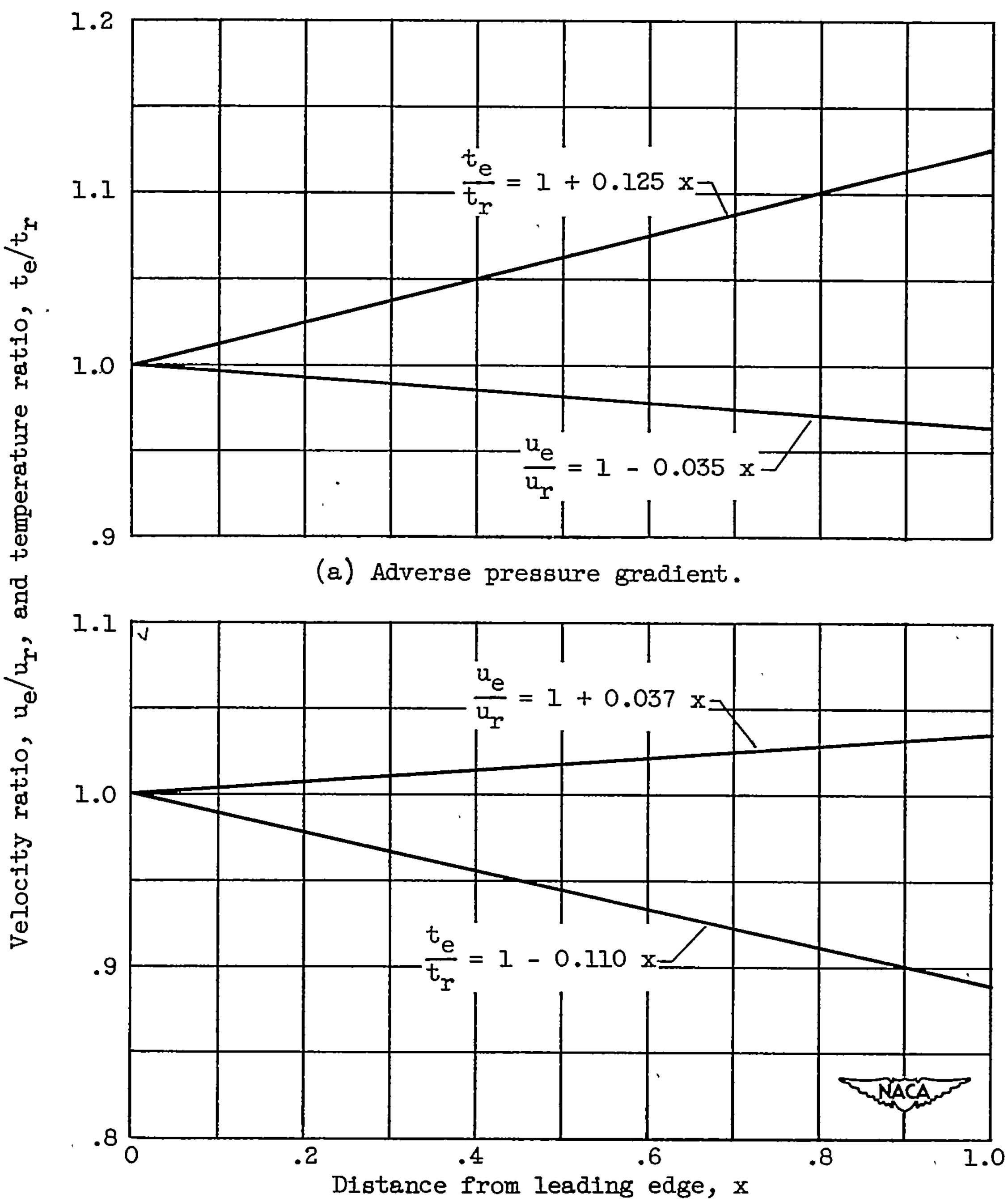
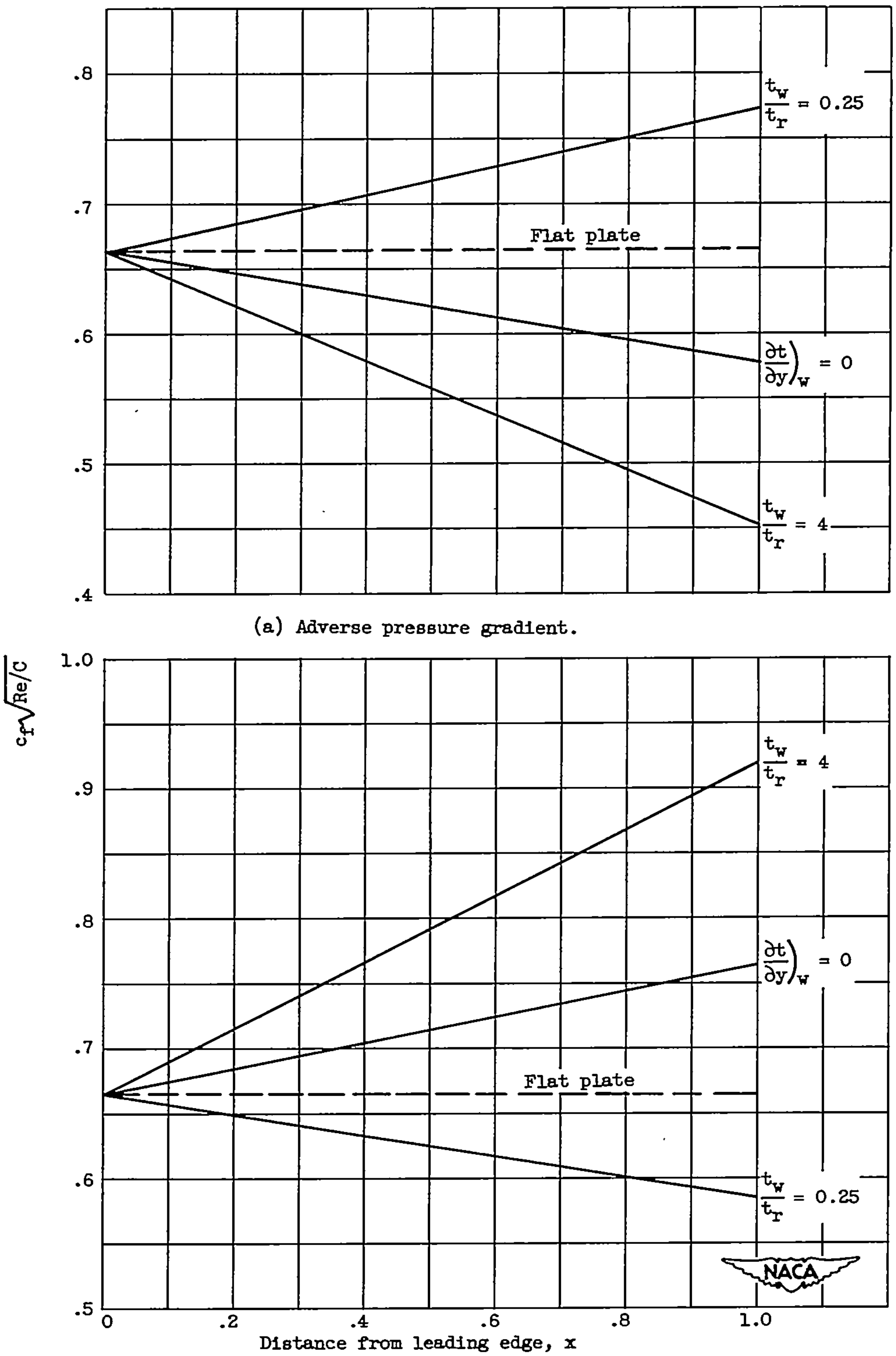
CR-8 back



(a) Adverse pressure gradient.

(b) Favorable pressure gradient.

Figure 5. - External velocity and temperature distributions on airfoils used for examples. M_∞, 3.



(a) Adverse pressure gradient.

(b) Favorable pressure gradient.

Figure 6. - Local skin friction as a function of distance from leading edge. M_∞, 3.

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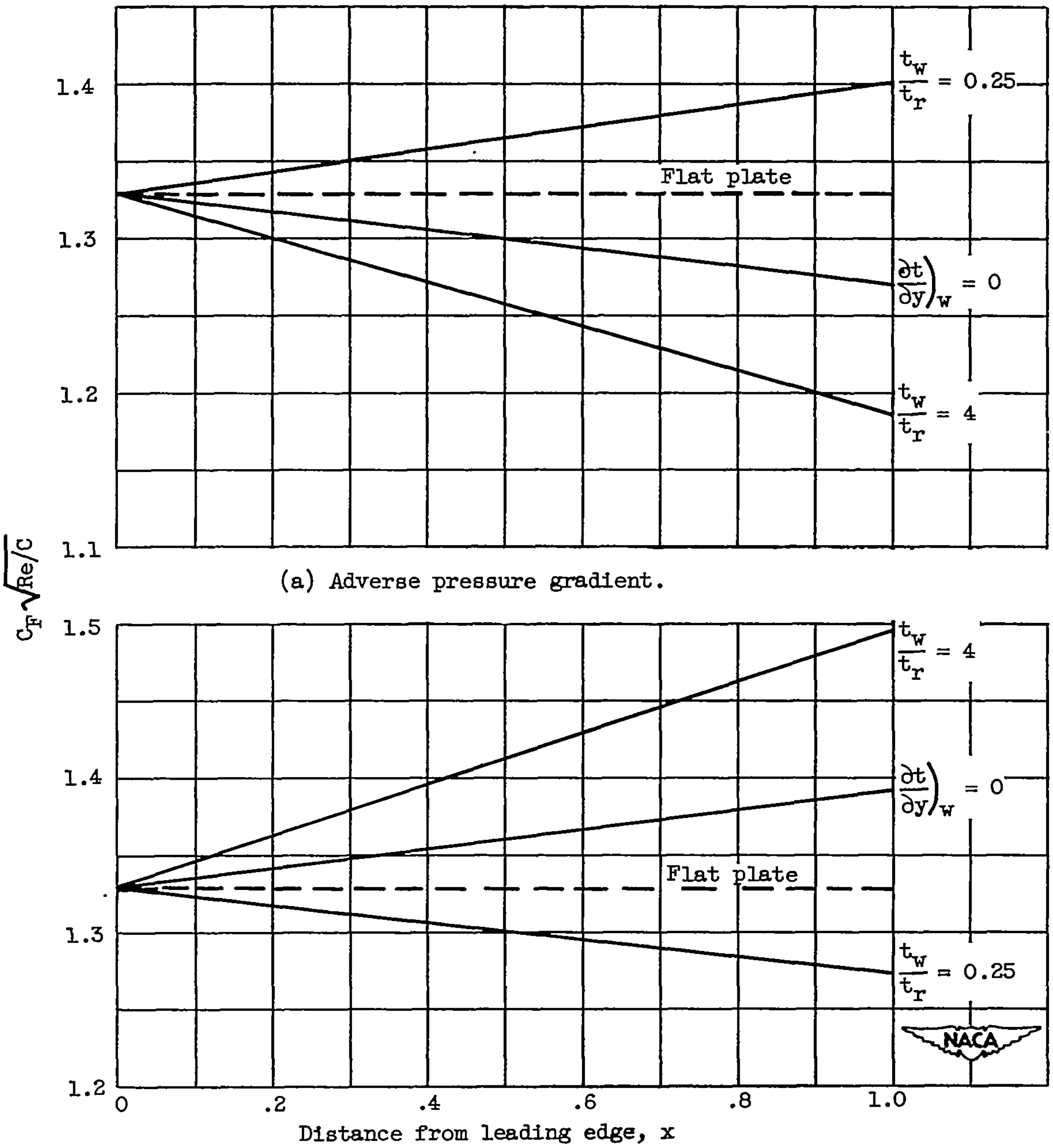

(a) Adverse pressure gradient.

(b) Favorable pressure gradient.

Figure 7. - Average friction drag as a function of distance from leading edge. M_∞, 3.

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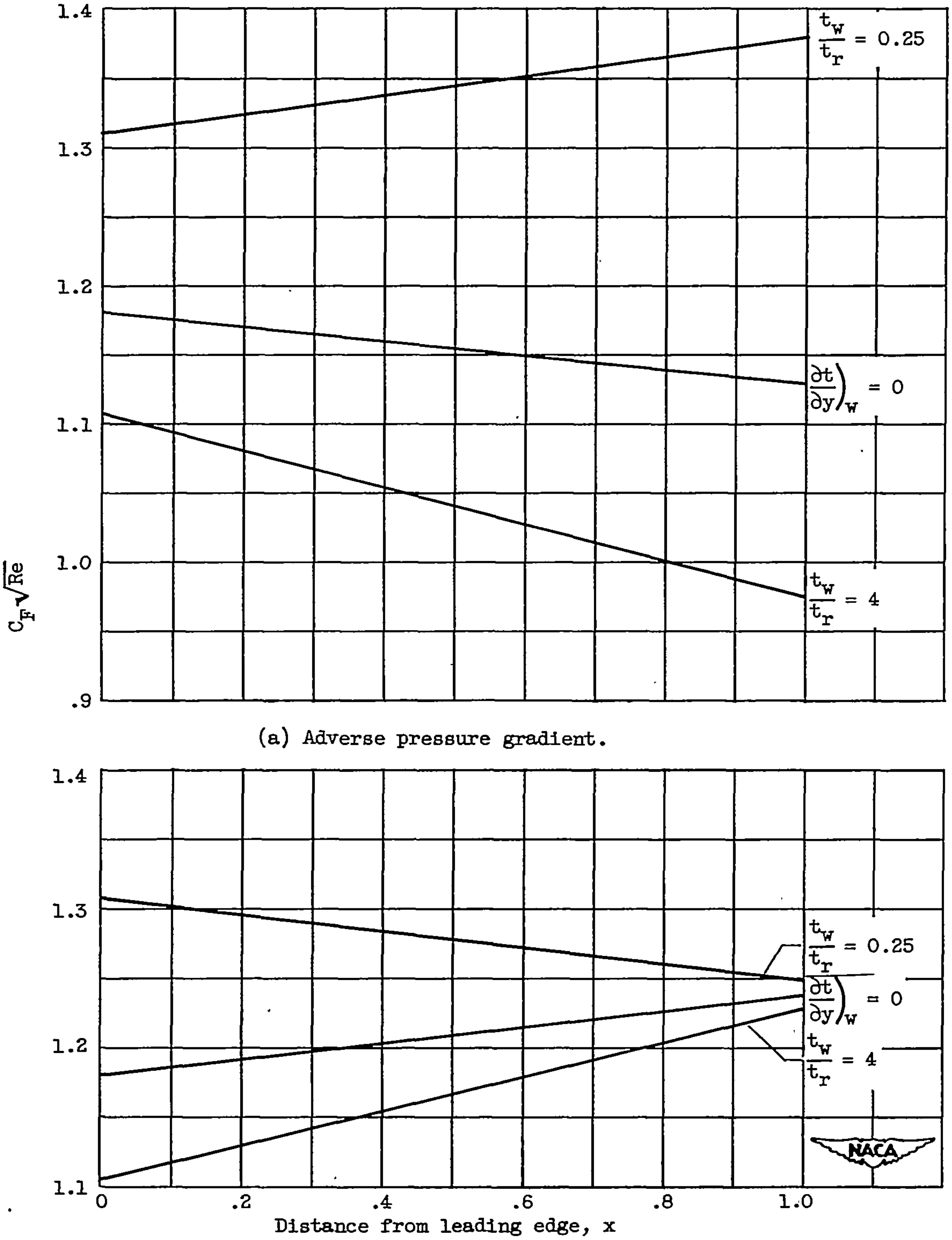

(b) Favorable pressure gradient.

Figure 8. - Average friction drag coefficient as a function of distance from leading edge. M_∞, 3; t_∞, -67° F.

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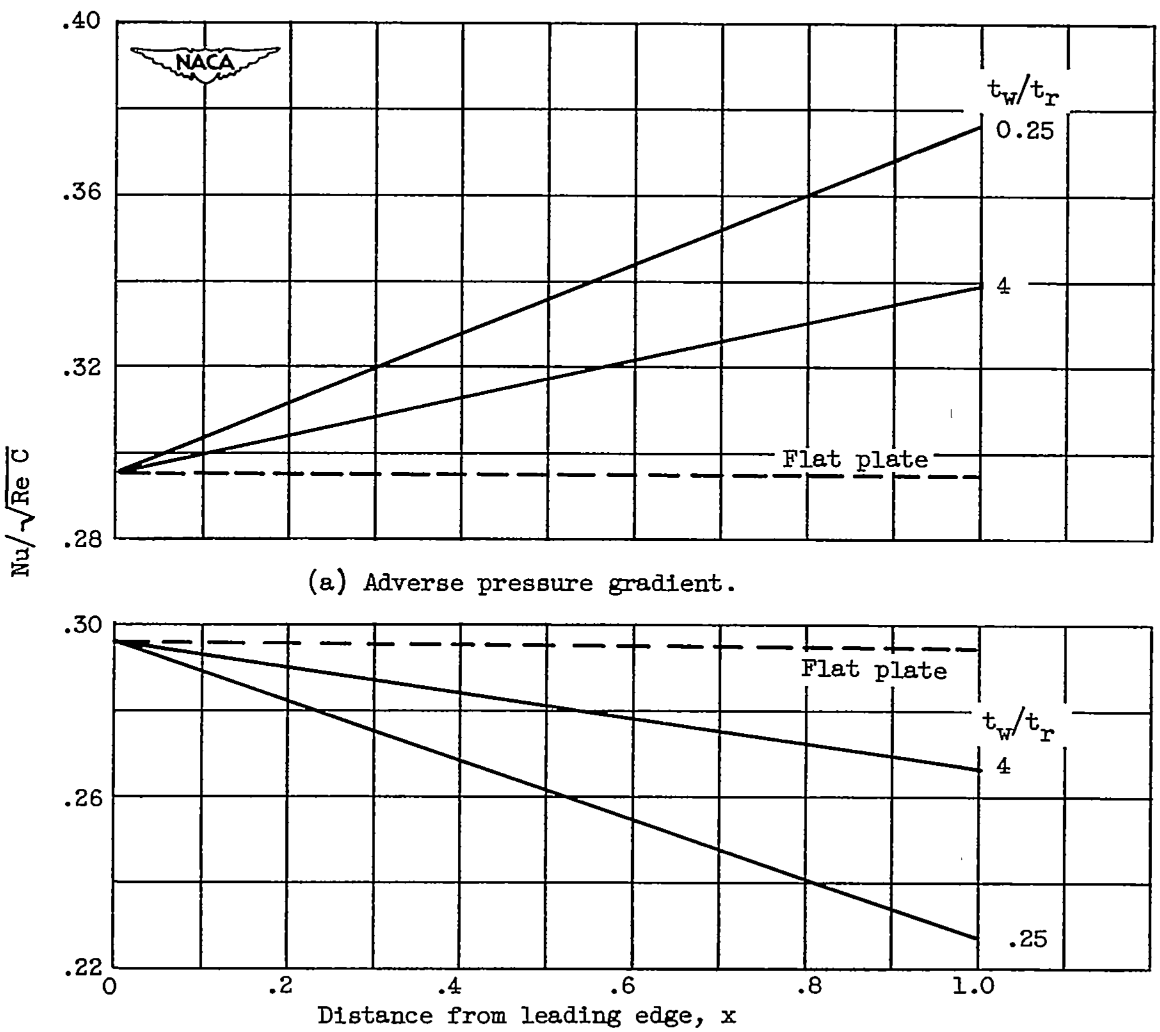
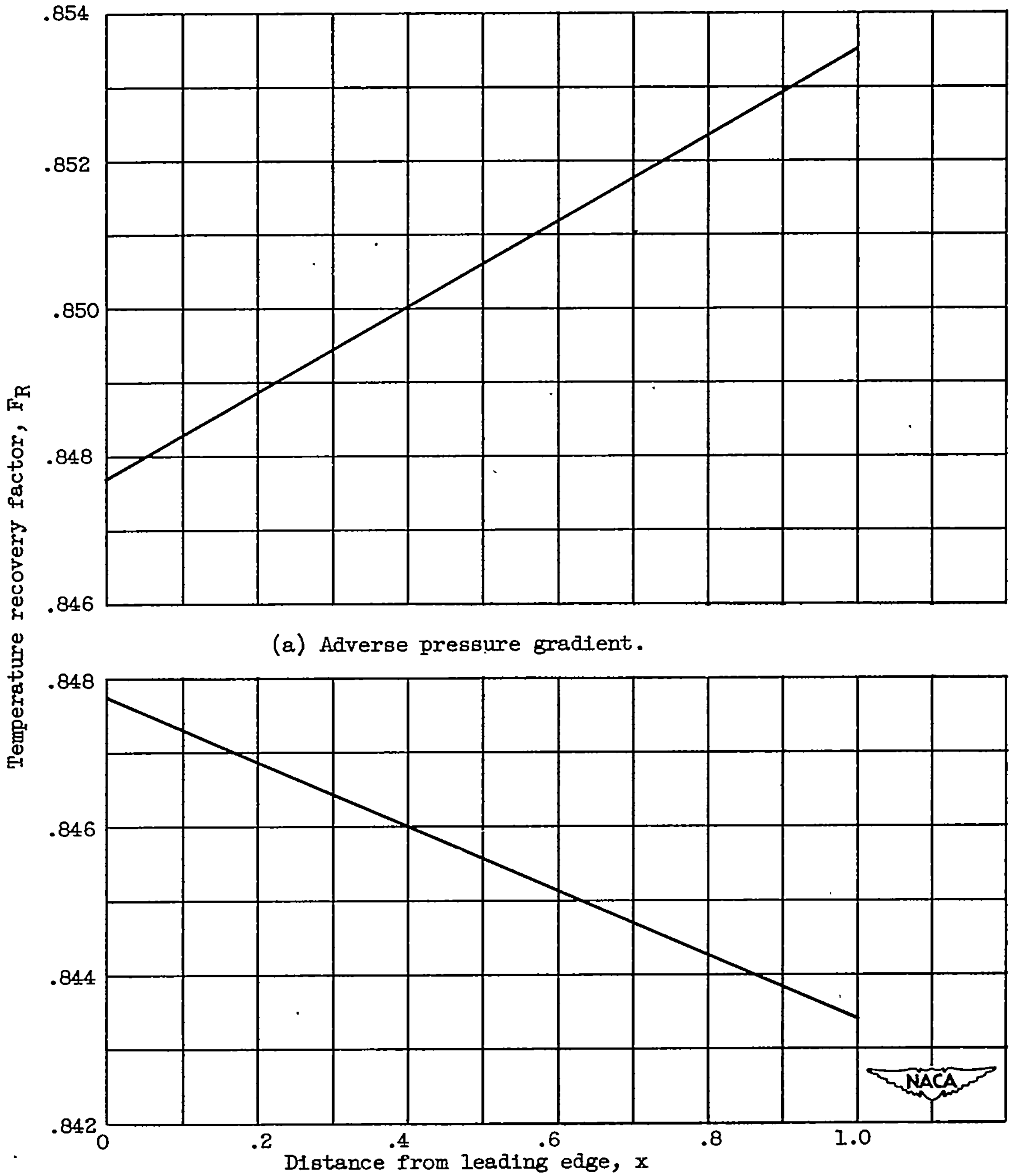


(a) Adverse pressure gradient.

(b) Favorable pressure gradient.

Figure 9. - Heat transfer as a function of distance from leading edge. M_∞, 3.



(b) Favorable pressure gradient.

Figure 10. - Temperature recovery factor as a function of distance from leading edge. M_∞, 3.

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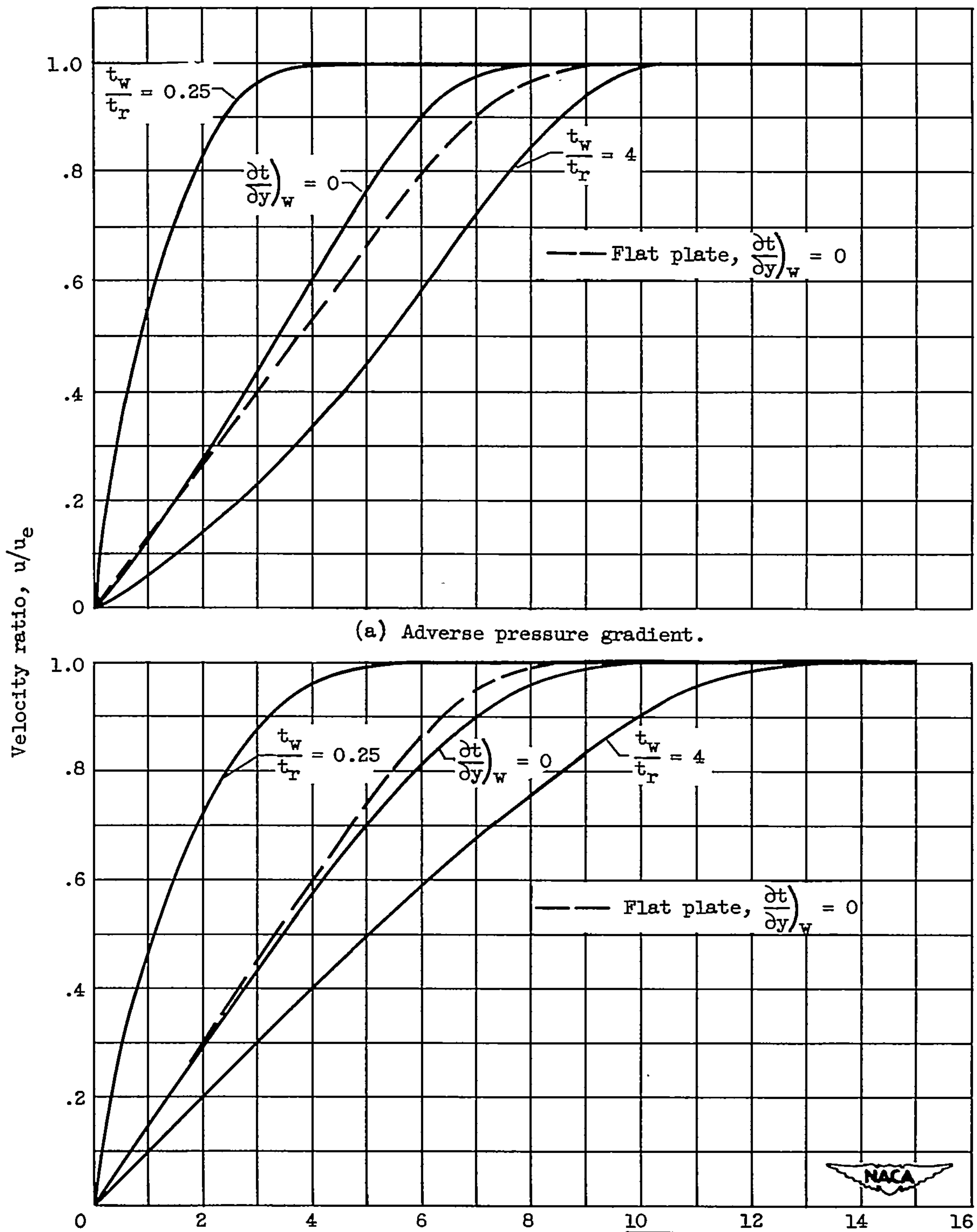
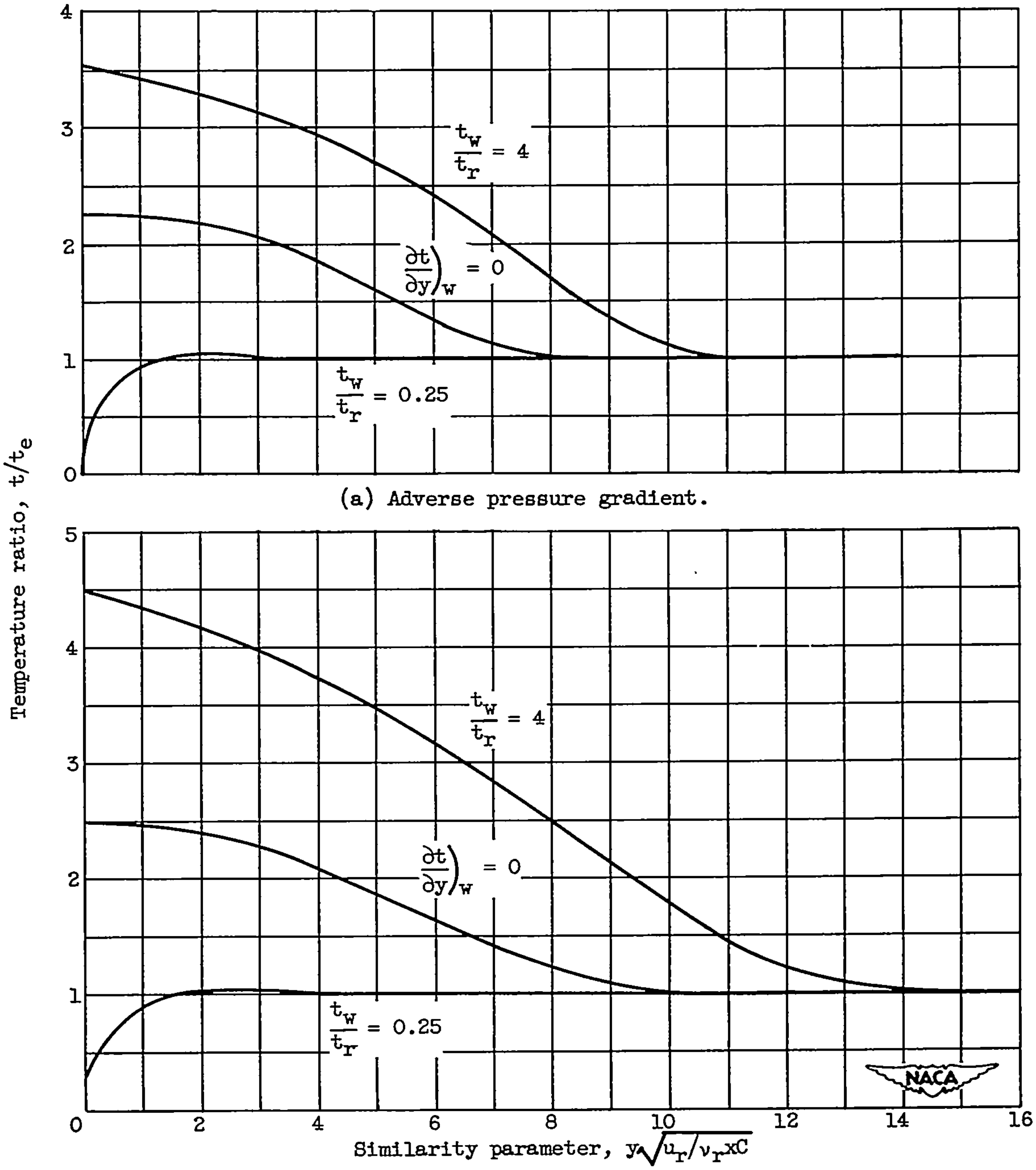


(a) Adverse pressure gradient.

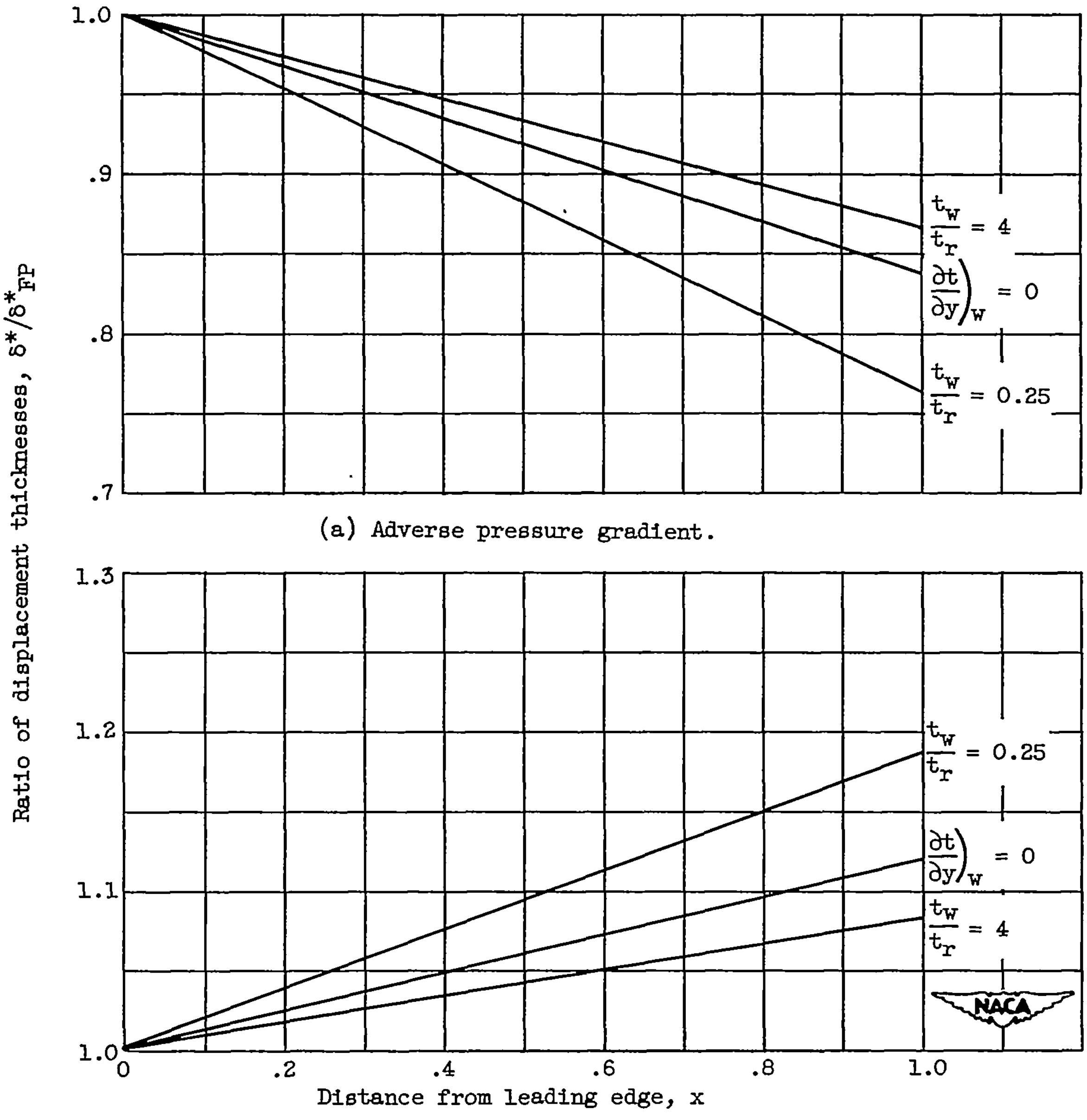
(b) Favorable pressure gradient.

Figure 11. - Velocity profiles. M_∞, 3; x, 1.



(b) Favorable pressure gradient.

Figure 12. - Temperature profiles. M_∞, 3; x, 1.



(b) Favorable pressure gradient.

Figure 13. - Ratio of displacement thicknesses as a function of distance from leading edge. M_∞, 3.

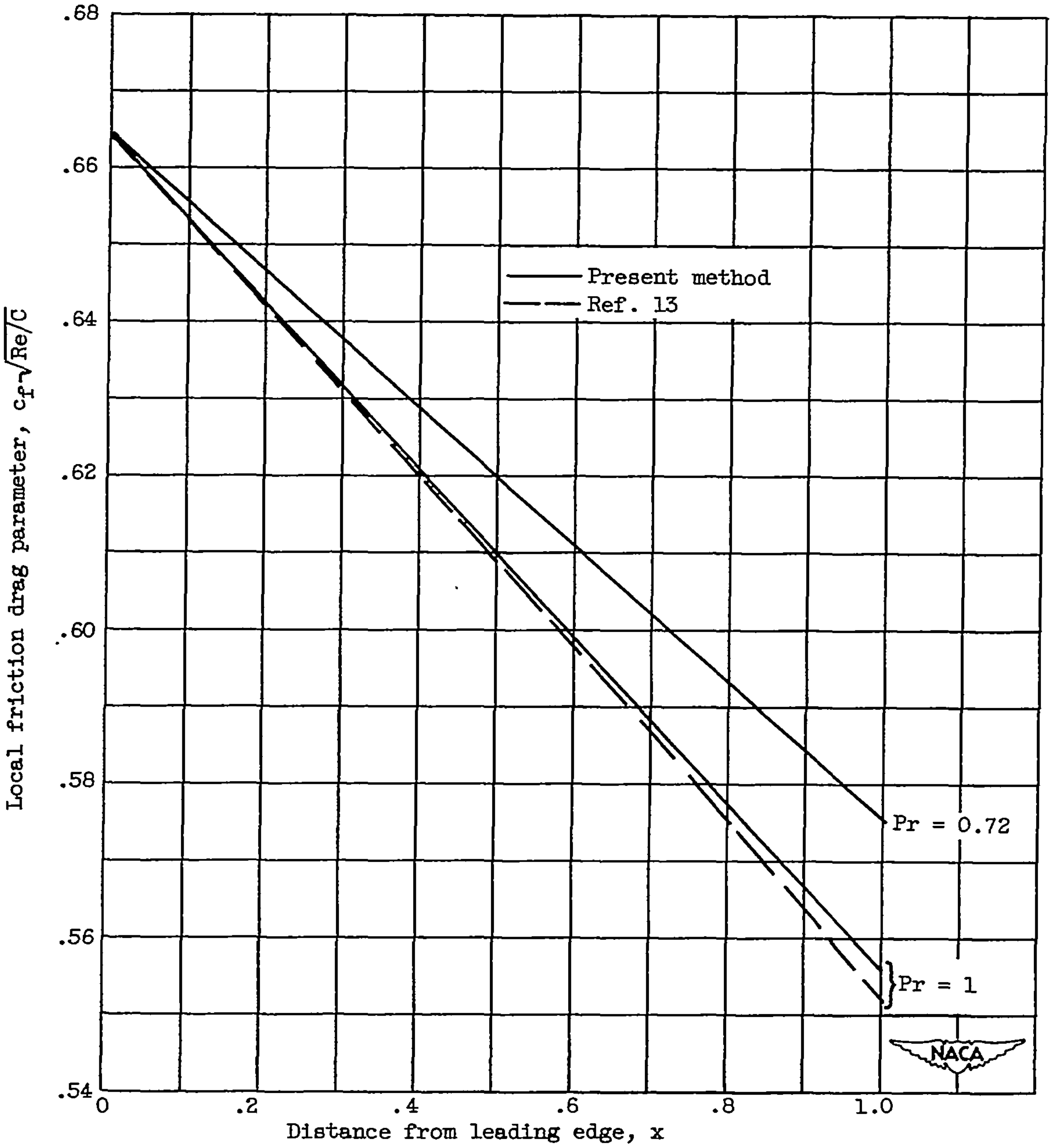


Figure 14. - Comparison of present results with results of reference 13. Zero heat transfer; adverse pressure gradient; M_∞, 3.

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